Eur. Phys. J. A **16**, 537–547 (2003)
DOI 10.1140/epja/i2002-10135-4

THE EUROPEAN PHYSICAL JOURNAL A

Covariant tensor formalism for partial-wave analyses of ψ decay to mesons

B.S. Zou[1,2,a] and D.V. Bugg[3]

[1] CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, PRC
[2] Institute of High Energy Physics, CAS, P.O. Box 918(4), Beijing 100039, PRC
[3] Queen Mary College, London, UK

Received: 29 November 2002 / Revised version: 19 December 2002 /
Published online: 25 March 2003 –
Communicated by V.V. Anisovich

Abstract. J/ψ and ψ' decays to mesons are a good place to look for glueballs, hybrids and for extracting strange and non-strange components in mesons. Abundant J/ψ and ψ' events have been collected at the Beijing Electron Positron Collider (BEPC). More data will be collected at upgraded BEPC and CLEO-C. Here we provide explicit PWA formulae for many interesting channels in the covariant tensor formalism.

PACS. 13.20.Gd Leptonic, semileptonic, and radiative decays of mesons: Decays of J/ψ, Υ, and other quarkonia – 13.25.Gv Hadronic decays of mesons: Decays of J/ψ, Υ, and other quarkonia – 13.66.Bc Hadron production in e^-e^+ interactions – 11.80.Et Partial-wave analysis

1 Introduction

High-statistics data have appeared from BES for J/ψ decays and will soon be available also for ψ' decays. Further high-statistics data are expected from CLEO [1]. It is convenient to have a uniform approach to partial-wave analyses. Here we provide one such approach using the covariant tensor formalism. A similar approach has been used in analyzing other reactions [2–4]. We provide formulae documenting those which have been used for a number of channels already published by BES [5–9] and extend them to further channels being prepared for publication. This list of reactions is not exhaustive, but formulae are readily extended to other cases following the same methods.

Reactions fall into two categories: non-radiative decays, where final-state particles are pions or kaons; all polarization information is then available in the form of angular distributions. Reactions of this type are discussed in sect. 2. This formalism extends also to final states containing ω, where polarization information is measured fully by the decay $\omega \to \pi^+\pi^-\pi^0$. The second class of reactions consists of radiative decays, *e.g.* $J/\psi \to \gamma\pi^+\pi^-$. For this class, differential cross-sections need to be summed over the unmeasured helicities of the photon, incorporating the knowledge that the photon is transverse. These reactions are considered in sect. 3.

[a] e-mail: `zoubs@mail.ihep.ac.cn`

2 Formalism for ψ non-radiative decay to mesons

The general form for the decay amplitude of a vector meson ψ with spin projection m is

$$A = \psi_\mu(m)A^\mu = \psi_\mu(m)\sum_i \Lambda_i U_i^\mu\,, \tag{1}$$

where $\psi_\mu(m)$ is the polarization vector of ψ; U_i^μ is the i-th partial-wave amplitude with coupling strength determined by a complex parameter Λ_i. The polarization vector satisfies

$$\sum_{m=1}^{3}\psi^\mu(m)\psi^{*\nu}(m) = -g^{\mu\nu} + \frac{p_\psi^\mu p_\psi^\nu}{p_\psi^2} \equiv -\tilde{g}^{\mu\nu}(p_\psi). \tag{2}$$

For ψ production from e^+e^- annihilation, the electrons are highly relativistic, with the result that $J_z = \pm 1$. If we take the beam direction to be the z-axis, this limits m to 1 and 2, *i.e.* components along x and y. Then the differential cross-section for the decay to an n-body final state is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_n} = \frac{(2\pi)^4}{2M_\psi}\cdot\frac{1}{2}\sum_{m=1}^{2}\psi_\mu(m)A^\mu\psi^*_{\mu'}(m)A^{*\mu'}\,, \tag{3}$$

where M_ψ is the mass of ψ and $\mathrm{d}\Phi_n$ is the standard element of the n-body phase space given by

$$\mathrm{d}\Phi_n(p_\psi;p_1,\cdots p_n) = \delta^4(p_\psi - \sum_{i=1}^{n}p_i)\prod_{i=1}^{n}\frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3 2E_i}. \tag{4}$$

$$B_1(Q_{abc}) = \sqrt{\frac{2}{Q_{abc}^2 + Q_0^2}}, \tag{14}$$

$$B_2(Q_{abc}) = \sqrt{\frac{13}{Q_{abc}^4 + 3Q_{abc}^2 Q_0^2 + 9Q_0^4}}, \tag{15}$$

$$B_3(Q_{abc}) = \sqrt{\frac{277}{Q_{abc}^6 + 6Q_{abc}^4 Q_0^2 + 45Q_{abc}^2 Q_0^4 + 225Q_0^6}}, \tag{16}$$

$$B_4(Q_{abc}) = \sqrt{\frac{12746}{Q_{abc}^8 + 10Q_{abc}^6 Q_0^2 + 135Q_{abc}^4 Q_0^4 + 1575Q_{abc}^2 Q_0^6 + 11025Q_0^8}}. \tag{17}$$

Note that

$$\sum_{m=1}^{2} \psi_\mu(m)\psi_{\mu'}^*(m) = \delta_{\mu\mu'}(\delta_{\mu 1} + \delta_{\mu 2}), \tag{5}$$

so we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_n} = \frac{1}{2}\sum_{\mu=1}^{2} A^\mu A^{*\mu} = \frac{1}{2}\sum_{i,j} \Lambda_i \Lambda_j^* \sum_{\mu=1}^{2} U_i^\mu U_j^{*\mu} \equiv \sum_{i,j} P_{ij} \cdot F_{ij}, \tag{6}$$

where

$$P_{ij} = P_{ji}^* = \Lambda_i \Lambda_j^*, \tag{7}$$

$$F_{ij} = F_{ji}^* = \frac{1}{2}\sum_{\mu=1}^{2} U_i^\mu U_j^{*\mu}. \tag{8}$$

We construct the partial-wave amplitudes U_i^μ in the covariant Rarita-Schwinger tensor formalism [10]. As in ref. [11], we use the pure-orbital-angular-momentum covariant tensors $\tilde{t}^{(l)}_{\mu_1\cdots\mu_l}$ and the covariant spin wave functions $\phi_{\mu_1\cdots\mu_S}$ together with the operators $g_{\mu\nu}$, $\epsilon_{\mu\nu\lambda\sigma}$ and the momenta of parent particles. For a process $a \to bc$, the covariant tensors $\tilde{t}^{(l)}_{\mu_1\cdots\mu_l}$ for final states of pure orbital angular momentum l are constructed from the relevant momenta p_a, p_b and p_c [11]:

$$\tilde{t}^{(0)} = 1, \tag{9}$$

$$\tilde{t}^{(1)}_\mu = \tilde{g}_{\mu\nu}(p_a) r^\nu B_1(Q_{abc}) \equiv \tilde{r}_\mu B_1(Q_{abc}), \tag{10}$$

$$\tilde{t}^{(2)}_{\mu\nu} = [\tilde{r}_\mu \tilde{r}_\nu - \frac{1}{3}(\tilde{r}\cdot\tilde{r})\tilde{g}_{\mu\nu}(p_a)] B_2(Q_{abc}), \tag{11}$$

$$\tilde{t}^{(3)}_{\mu\nu\lambda} = [\tilde{r}_\mu \tilde{r}_\nu \tilde{r}_\lambda - \frac{1}{5}(\tilde{r}\cdot\tilde{r})(\tilde{g}_{\mu\nu}(p_a)\tilde{r}_\lambda + \tilde{g}_{\nu\lambda}(p_a)\tilde{r}_\mu + \tilde{g}_{\lambda\mu}(p_a)\tilde{r}_\nu)] B_3(Q_{abc}), \tag{12}$$

$$\cdots$$

with $r = p_b - p_c$. The term $(\tilde{r}\cdot\tilde{r})$ is the dot-product of 4-vectors: $\tilde{r}_0\tilde{r}_0 - \tilde{r}_1\tilde{r}_1 - \tilde{r}_2\tilde{r}_2 - \tilde{r}_3\tilde{r}_3$, and makes $\tilde{t}^{(2)}_{\mu\nu}$ traceless. Likewise $\tilde{t}^{(3)}$ is constructed to be traceless. Q_{abc} is the magnitude of $\mathbf{p}_b$ or $\mathbf{p}_c$ in the rest system of a, where

$$Q_{abc}^2 = \frac{(s_a + s_b - s_c)^2}{4s_a} - s_b \tag{13}$$

with $s_a = E_a^2 - \mathbf{p}_a^2$. Then $\tilde{t}^{(l)}_{\mu_1\cdots\mu_l}$ contains the angular distribution function multiplied by a Blatt-Weisskopf barrier factor [11,12] $Q_{abc}^l B_l(Q_{abc})$. Explicitly

see equations (14)–(17) *above*

Here Q_0 is a hadron "scale" parameter $Q_0 = 0.197321/R$ GeV/c, where R is the radius of the centrifugal barrier in fm. We remark that in these Blatt-Weisskopf factors, the approximation is made that the centrifugal barrier may be replaced by a square well of radius R.

If a is an intermediate resonance decaying into bc, one needs to introduce into the amplitude a Breit-Wigner propagator denoted by $f^{(a)}_{(bc)}$:

$$f^{(a)}_{(bc)} = \frac{1}{m_a^2 - s_{bc} - i m_a \Gamma_a}; \tag{18}$$

here $s_{bc} = (p_b + p_c)^2$ is the invariant mass-squared of b and c; m_a, Γ_a are the resonance mass and width.

We outline now some further general features of notation, taking as an example the two-step process $J/\psi \to \rho_{12}\pi_3$, $\rho_{12} \to \pi_1\pi_2$. In the first step we denote the orbital angular momentum by L; in this example $L = 1$. In the second step, we denote the orbital angular momentum by ℓ, which is again 1 in this case. The tensor describing the first step will be denoted by $\tilde{T}^{(L)}_{\mu_1\cdots\mu_L}$. The tensor describing the second step will be denoted by $\tilde{t}^{(l)}_{\mu_1\cdots\mu_l}$. The orbital angular momentum is constructed in terms of relative momenta, so it is convenient to define $q_{(ij)} = p_i - p_j$.

Some expressions depend also on the total momentum of the ij pair: $p_{(ij)} = p_i + p_j$. When one wants to combine two angular momenta ($\mathbf{j}_b$ and $\mathbf{j}_c$) into a total angular momentum $\mathbf{j}_a$, if $j_a + j_b + j_c$ is an odd number, then a combination $\epsilon_{\mu\nu\lambda\sigma} p_a^\mu$, with p_a the momentum of the parent particle, is needed; otherwise it is not needed.

Projection operators will be a useful general tool in constructing expressions. For a meson a with spin S and corresponding spin wave function $\phi_{\mu_1\cdots\mu_S}(p_a, m)$, what we usually need to use in constructing amplitudes is its spin projection operator $P^{(S)}_{\mu_1\cdots\mu_S\mu'_1\cdots\mu'_S}(p_a)$:

$$P^{(1)}_{\mu\mu'}(p_a)=\sum_m \phi_\mu(p_a,m)\phi^*_{\mu'}(p_a,m) = -g_{\mu\mu'}+\frac{p_{a\mu}p_{a\mu'}}{p_a^2}\equiv -\tilde{g}_{\mu\mu'}(p_a), \quad (19)$$

$$P^{(2)}_{\mu\nu\mu'\nu'}(p_a)=\sum_m \phi_{\mu\nu}(p_a,m)\phi^*_{\mu'\nu'}(p_a,m) = \frac{1}{2}(\tilde{g}_{\mu\mu'}\tilde{g}_{\nu\nu'}+\tilde{g}_{\mu\nu'}\tilde{g}_{\nu\mu'})-\frac{1}{3}\tilde{g}_{\mu\nu}\tilde{g}_{\mu'\nu'}, \quad (20)$$

$$\begin{aligned}P^{(3)}_{\mu\nu\lambda\mu'\nu'\lambda'}(p_a)=&\sum_m \phi_{\mu\nu\lambda}(p_a,m)\phi^*_{\mu'\nu'\lambda'}(p_a,m) = \\ &-\frac{1}{6}(\tilde{g}_{\mu\mu'}\tilde{g}_{\nu\nu'}\tilde{g}_{\lambda\lambda'}+\tilde{g}_{\mu\mu'}\tilde{g}_{\nu\lambda'}\tilde{g}_{\lambda\nu'}+\tilde{g}_{\mu\nu'}\tilde{g}_{\nu\mu'}\tilde{g}_{\lambda\lambda'} \\ &+\tilde{g}_{\mu\nu'}\tilde{g}_{\nu\lambda'}\tilde{g}_{\lambda\mu'}+\tilde{g}_{\mu\lambda'}\tilde{g}_{\nu\nu'}\tilde{g}_{\lambda\mu'}+\tilde{g}_{\mu\lambda'}\tilde{g}_{\nu\mu'}\tilde{g}_{\lambda\nu'}) \\ &+\frac{1}{15}(\tilde{g}_{\mu\nu}\tilde{g}_{\mu'\nu'}\tilde{g}_{\lambda\lambda'}+\tilde{g}_{\mu\nu}\tilde{g}_{\nu'\lambda'}\tilde{g}_{\lambda\mu'}+\tilde{g}_{\mu\nu}\tilde{g}_{\mu'\lambda'}\tilde{g}_{\lambda\nu'} \\ &+\tilde{g}_{\mu\lambda}\tilde{g}_{\mu'\lambda'}\tilde{g}_{\nu\nu'}+\tilde{g}_{\mu\lambda}\tilde{g}_{\mu'\nu'}\tilde{g}_{\nu\lambda'}+\tilde{g}_{\mu\lambda}\tilde{g}_{\nu'\lambda'}\tilde{g}_{\nu\mu'} \\ &+\tilde{g}_{\nu\lambda}\tilde{g}_{\nu'\lambda'}\tilde{g}_{\mu\mu'}+\tilde{g}_{\nu\lambda}\tilde{g}_{\mu'\nu'}\tilde{g}_{\mu\lambda'}+\tilde{g}_{\nu\lambda}\tilde{g}_{\mu'\lambda'}\tilde{g}_{\mu\nu'}),\end{aligned} \quad (21)$$

$$\begin{aligned}P^{(4)}_{\mu\nu\lambda\sigma\mu'\nu'\lambda'\sigma'}(p_a)=&\sum_m \phi_{\mu\nu\lambda\sigma}(p_a,m)\phi^*_{\mu'\nu'\lambda'\sigma'}(p_a,m) = \\ &\frac{1}{24}[\tilde{g}_{\mu\mu'}\tilde{g}_{\nu\nu'}\tilde{g}_{\lambda\lambda'}\tilde{g}_{\sigma\sigma'}+\cdots \\ &\quad(\mu',\nu',\lambda',\sigma' \text{ permutation, 24 terms})] \\ &-\frac{1}{84}[\tilde{g}_{\mu\nu}\tilde{g}_{\mu'\nu'}\tilde{g}_{\lambda\lambda'}\tilde{g}_{\sigma\sigma'}+\cdots \\ &\quad(\mu,\nu,\lambda,\sigma \text{ permutation,} \\ &\quad\quad\mu',\nu',\lambda',\sigma' \text{ permutation, 72 terms})] \\ &+\frac{1}{105}(\tilde{g}_{\mu\nu}\tilde{g}_{\lambda\sigma}+\tilde{g}_{\mu\lambda}\tilde{g}_{\nu\sigma}+\tilde{g}_{\mu\sigma}\tilde{g}_{\nu\lambda}) \\ &\quad\cdot(\tilde{g}_{\mu'\nu'}\tilde{g}_{\lambda'\sigma'}+\tilde{g}_{\mu'\lambda'}\tilde{g}_{\nu'\sigma'}+\tilde{g}_{\mu'\sigma'}\tilde{g}_{\nu'\lambda'}).\end{aligned} \quad (22)$$

Note that

$$\tilde{t}^{(L)}_{\mu_1\cdots\mu_L}=(-1)^L P^{(L)}_{\mu_1\cdots\mu_L\mu'_1\cdots\mu'_L} r^{\mu'_1}\cdots r^{\mu'_L} B_L(Q_{abc}). \quad (23)$$

We come now to specific examples of reactions.

2.1 $\psi \to \pi^+\pi^-\pi^0$

For three isospin-1 particles coupling to an isospin-zero particle, the only possible coupling for isospin conservation is $(\mathbf{I}_1\times\mathbf{I}_2)\cdot\mathbf{I}_3$, which is fully anti-symmetric in particles 1, 2, 3. This demands that the angular dependent part should also be fully anti-symmetric for particles 1, 2, 3, in order to make the overall amplitude symmetric. For $\psi\to\pi^+\pi^-\pi^0$, any two pions are limited to an overall isospin 1 and hence can only be negative-parity states with J odd, *i.e.*, $J^P=1^-$, 3^-, 5^- etc.

For $\psi\to\rho(1^-)\pi\to\pi^+\pi^-\pi^0$, ψ decays to $\rho\pi$ in a P-wave; then ρ decays to $\pi\pi$ also in a P-wave, hence the amplitude for the two-step process is

$$\begin{aligned}U^\mu_\rho=&(\mathbf{I}_1\times\mathbf{I}_2)\cdot\mathbf{I}_3\ \epsilon_{\mu\nu\lambda\sigma}p^\sigma_\psi\tilde{T}^{(1)\nu}_{(\rho 3)}\tilde{t}^{(1)\lambda}_{(12)}f^{(\rho)}_{(12)} \\ &+(1\leftrightarrow 3)+(2\leftrightarrow 3) \\ =&\,4i\epsilon_{\mu\nu\lambda\sigma}p^\nu_1p^\lambda_2p^\sigma_3\Big[B_1(Q_{\psi\rho 3})f^{(\rho)}_{(12)}B_1(Q_{\rho 12}) \\ &+B_1(Q_{\psi\rho 2})f^{(\rho)}_{(13)}B_1(Q_{\rho 13}) \\ &+B_1(Q_{\psi\rho 1})f^{(\rho)}_{(23)}B_1(Q_{\rho 23})\Big].\end{aligned} \quad (24)$$

Here we use the convention $\mathbf{I}_1=(\frac{-1}{\sqrt{2}},\frac{-i}{\sqrt{2}},0)$ for π^+, $\mathbf{I}_2=(\frac{1}{\sqrt{2}},\frac{-i}{\sqrt{2}},0)$ for π^- and $\mathbf{I}_3=(0,0,1)$ for π^0. This gives $(\mathbf{I}_1\times\mathbf{I}_2)\cdot\mathbf{I}_3=-i$.

The amplitude can be further simplified in the ψ rest system as

$$\begin{aligned}U^\mu_\rho=&\,4iM_\psi\epsilon_{\mu\nu\lambda 0}p^\nu_1p^\lambda_2\Big[B_1(Q_{\psi\rho 3})f^{(\rho)}_{(12)}B_1(Q_{\rho 12}) \\ &+B_1(Q_{\psi\rho 2})f^{(\rho)}_{(13)}B_1(Q_{\rho 13}) \\ &+B_1(Q_{\psi\rho 1})f^{(\rho)}_{(23)}B_1(Q_{\rho 23})\Big].\end{aligned} \quad (25)$$

For any other 1^- intermediate state ρ', one can get the corresponding amplitude by simply replacing the Breit-Wigner component $f^{(\rho)}$ by $f^{(\rho')}$.

For $\psi\to\rho_3(3^-)\pi\to\pi^+\pi^-\pi^0$, ψ decays to $\rho_3\pi$ in an F-wave; then ρ_3 decays to $\pi\pi$ also in an F-wave; the amplitude is

$$\begin{aligned}U^\mu_{\rho_3}=&(\mathbf{I}_1\times\mathbf{I}_2)\cdot\mathbf{I}_3\ \epsilon_{\mu\nu\lambda\sigma}p^\sigma_\psi\tilde{T}^{(3)\nu\alpha\beta}_{(\rho_3 3)}\tilde{t}^{(3)\lambda}_{(12)\alpha\beta}\cdot f^{(\rho_3)}_{(12)} \\ &+(1\leftrightarrow 3)+(2\leftrightarrow 3) \\ =&-i\epsilon_{\mu\nu\lambda\sigma}p^\sigma_\psi[\tilde{T}^{(3)\nu\alpha\beta}_{(\rho_3 3)}\tilde{t}^{(3)\lambda}_{(12)\alpha\beta}\cdot f^{(\rho_3)}_{(12)} \\ &-(1\leftrightarrow 3)-(2\leftrightarrow 3)].\end{aligned} \quad (26)$$

Similarly, for $\psi\to\rho_5(5^-)\pi\to\pi^+\pi^-\pi^0$, the amplitude should be

$$\begin{aligned}U^\mu_{\rho_5}=&(\mathbf{I}_1\times\mathbf{I}_2)\cdot\mathbf{I}_3\ \epsilon_{\mu\nu\lambda\sigma}p^\sigma_\psi\tilde{T}^{(5)\nu\alpha\beta\gamma\delta}_{(\rho_5 3)}\tilde{t}^{(5)\lambda}_{(12)\alpha\beta\gamma\delta}f^{(\rho_5)}_{(12)} \\ &+(1\leftrightarrow 3)+(2\leftrightarrow 3) \\ =&-i\epsilon_{\mu\nu\lambda\sigma}p^\sigma_\psi[\tilde{T}^{(5)\nu\alpha\beta\gamma\delta}_{(\rho_5 3)}\tilde{t}^{(5)\lambda}_{(12)\alpha\beta\gamma\delta}f^{(\rho_5)}_{(12)} \\ &-(1\leftrightarrow 3)-(2\leftrightarrow 3)].\end{aligned} \quad (27)$$

If one considers a small isospin symmetry-breaking effect, a free parameter can be multiplied by the term corresponding to the ρ^0 intermediate state.

2.2 $\psi \to K^+K^-\pi^0$

This channel is similar to $\pi^+\pi^-\pi^0$. However, we now need to consider resonances for both $K\pi$ and K^+K^-

subsystems. Numbering K^+, K^-, π^0 as particle 1, 2, 3, the possible partial-wave amplitudes are the following:

$$U^{\mu}_{\rho'} = \epsilon_{\mu\nu\lambda\sigma} p_1^{\nu} p_2^{\lambda} p_3^{\sigma} B_1(Q_{\psi\rho' 3}) f^{(\rho')}_{(12)} B_1(Q_{\rho' 12}), \quad (28)$$

$$U^{\mu}_{\rho_3} = \epsilon_{\mu\nu\lambda\sigma} p_{\psi}^{\sigma} \tilde{T}^{(3)\nu\alpha\beta}_{(\rho_3 3)} \tilde{t}^{(3)\lambda}_{(12)\alpha\beta} \cdot f^{(\rho_3)}_{(12)}, \quad (29)$$

$$U^{\mu}_{\rho_5} = \epsilon_{\mu\nu\lambda\sigma} p_{\psi}^{\sigma} \tilde{T}^{(5)\nu\alpha\beta\gamma\delta}_{(\rho_5 3)} \tilde{t}^{(5)\lambda}_{(12)\alpha\beta\gamma\delta} f^{(\rho_5)}_{(12)}, \quad (30)$$

$$U^{\mu}_{K^*} = \epsilon_{\mu\nu\lambda\sigma} p_1^{\nu} p_2^{\lambda} p_3^{\sigma} \Big[B_1(Q_{\psi K^* 2}) f^{(K^*)}_{(13)} B_1(Q_{K^* 13}) + B_1(Q_{\psi K^* 1}) f^{(K^*)}_{(23)} B_1(Q_{K^* 23}) \Big], \quad (31)$$

$$U^{\mu}_{K_3^*} = \epsilon_{\mu\nu\lambda\sigma} p_{\psi}^{\sigma} \Big[\tilde{T}^{(3)\nu\alpha\beta}_{(K_3^* 2)} \tilde{t}^{(3)\lambda}_{(13)\alpha\beta} \cdot f^{(K_3^*)}_{(13)} - (1 \leftrightarrow 2) \Big], \quad (32)$$

$$U^{\mu}_{K_5^*} = \epsilon_{\mu\nu\lambda\sigma} p_{\psi}^{\sigma} \Big[\tilde{T}^{(5)\nu\alpha\beta\gamma\delta}_{(K_5^* 2)} \tilde{t}^{(5)\lambda}_{(13)\alpha\beta\gamma\delta} f^{(K_5^*)}_{(13)} - (1 \leftrightarrow 2) \Big]. \quad (33)$$

2.3 $\psi \to \phi\pi^+\pi^- \to K^+K^-\pi^+\pi^-$

For this channel, ϕ is reconstructed from two kaons; most possible intermediate states are ϕ plus an isospin-zero resonance, f_0 or f_2, which decays into two pions. The f_4 is unlikely to be produced, because the ψ mass is not far from the ϕf_4 threshold and the decay requires $L = 2$ between ϕ and f_4, hence a strong centrifugal barrier. For $\psi \to \phi f_J$ in an orbital-angular-momentum L state, the conservation of the total angular momentum requires

$$\mathbf{S}_{\psi} = \mathbf{S} + \mathbf{L}, \quad (34)$$

where

$$\mathbf{S} = \mathbf{S}_{\phi} + \mathbf{J}. \quad (35)$$

In the following, we use the notation $\langle \phi f_J | LS \rangle$ to denote the corresponding partial-wave amplitude U^{μ}_i. We number the K^+, K^-, π^+, π^- as particle 1, 2, 3, 4, respectively. Then we have two independent partial-wave amplitudes for each f_0 production. In the general formalism, they may be written

$$\langle \phi f_0 | 01 \rangle = \tilde{t}^{(1)\mu}_{(12)} f^{(\phi)}_{(12)} f^{(f_0)}_{(34)}, \quad (36)$$

$$\langle \phi f_0 | 21 \rangle = \tilde{T}^{(2)\mu\nu}_{(\phi f_0)} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_0)}_{(34)}. \quad (37)$$

For the very narrow ϕ resonance, the $\ell = 1$ centrifugal barrier factor for the ϕ decay has a negligible effect on the ϕ line-shape and can be dropped. The expression for $t^{(1)\mu}_{(12)}$ simplifies to

$$t^{(1)\mu}_{(12)} = \tilde{r}^{\mu} = q^{\mu}_{(12)}.$$

In the last step, we use the fact that K^+ and K^- have equal masses. Then eqs. (36) and (37) become

$$\langle \phi f_0 | 01 \rangle = q^{\mu}_{(12)} f^{(\phi)}_{(12)} f^{(f_0)}_{(34)}, \quad (38)$$

$$\langle \phi f_0 | 21 \rangle = \tau^{\mu\nu} q_{(12)\nu} B_2(Q_{\psi\phi f_0}) f^{(\phi)}_{(12)} f^{(f_0)}_{(34)}, \quad (39)$$

where $\tau^{\mu\nu}$ is the $L = 2$ operator,

$$\tau^{\mu\nu} = q^{\mu}_{(12)} q^{\nu}_{(12)} - \frac{1}{3}(q_{(12)} \cdot q_{(12)}) g^{\mu\nu}. \quad (40)$$

For each f_2 production, there are five independent partial waves, which we retain in their general form:

$$\langle \phi f_2 | 01 \rangle = \tilde{t}^{(2)\mu\nu}_{(34)} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_2)}_{(34)}, \quad (41)$$

$$\langle \phi f_2 | 21 \rangle = \tilde{T}^{(2)\mu\alpha}_{(\phi f_2)} \tilde{t}^{(2)}_{(34)\alpha\nu} \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(f_2)}_{(34)}, \quad (42)$$

$$\langle \phi f_2 | 22 \rangle = \epsilon^{\mu\alpha\beta\gamma} p_{\psi\alpha} \tilde{T}^{(2)\delta}_{(\phi f_2)\beta} [\epsilon_{\gamma\lambda\sigma\nu} \tilde{t}^{(2)\lambda}_{(34)\delta} + \epsilon_{\delta\lambda\sigma\nu} \tilde{t}^{(2)\lambda}_{(34)\gamma}] p^{\sigma}_{\psi} \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(f_2)}_{(34)}, \quad (43)$$

$$\langle \phi f_2 | 23 \rangle = P^{(3)\mu\alpha\beta\gamma\delta\nu}(p_{\psi}) \tilde{T}^{(2)}_{(\phi f_2)\alpha\beta} \tilde{t}^{(2)}_{(34)\gamma\delta} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_2)}_{(34)}, \quad (44)$$

$$\langle \phi f_2 | 43 \rangle = \tilde{T}^{(4)\mu\nu\lambda\sigma}_{(\phi f_2)} \tilde{t}^{(1)}_{(12)\nu} \tilde{t}^{(2)}_{(34)\lambda\sigma} f^{(\phi)}_{(12)} f^{(f_2)}_{(34)}. \quad (45)$$

There is no established resonance decaying into $\phi\pi$. However, there are speculations about $(s\bar{s}q\bar{q})$ four-quark states which could decay to $\phi\pi$. So, here we also give some partial-wave amplitudes for $\psi \to X\pi$ with the intermediate resonance X further decaying to $\phi\pi$. For X being a $\rho'(1^{--})$ state, there is only one independent amplitude since both $\psi \to \rho'\pi$ and $\rho' \to \phi\pi$ are limited to a P-wave:

$$U^{\mu}_{\rho'} = \epsilon^{\mu}_{\alpha\beta\gamma} p^{\alpha}_{\psi} \Big[\tilde{T}^{(1)\beta}_{(\rho' 3)} \epsilon^{\gamma\delta\sigma\lambda} p_{\psi\delta} \tilde{t}^{(1)}_{(\phi 4)\sigma} \tilde{t}^{(1)}_{(12)\lambda} f^{(\phi)}_{(12)} f^{(\rho')}_{(\phi 4)} + \tilde{T}^{(1)\beta}_{(\rho' 4)} \epsilon^{\gamma\delta\sigma\lambda} p_{\psi\delta} \tilde{t}^{(1)}_{(\phi 3)\sigma} \tilde{t}^{(1)}_{(12)\lambda} f^{(\phi)}_{(12)} f^{(\rho')}_{(\phi 3)} \Big]. \quad (46)$$

For X being a $b_1(1^{+-})$ state, there are four independent amplitudes since both $\psi \to b_1\pi$ and $b_1 \to \phi\pi$ can have both S and D waves:

$$U^{\mu}_{b_1 SS} = \tilde{g}^{\mu\nu}_{(123)} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(b_1)}_{(123)} + \tilde{g}^{\mu\nu}_{(124)} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(b_1)}_{(124)}, \quad (47)$$

$$U^{\mu}_{b_1 SD} = \tilde{t}^{(2)\mu\nu}_{(\phi 3)} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(b_1)}_{(123)} + \tilde{t}^{(2)\mu\nu}_{(\phi 4)} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(b_1)}_{(124)}, \quad (48)$$

$$U^{\mu}_{b_1 DS} = \tilde{T}^{(2)\mu\lambda}_{(b_1 4)} \tilde{g}_{(123)\lambda\nu} \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(b_1)}_{(123)} + \tilde{T}^{(2)\mu\lambda}_{(b_1 3)} \tilde{g}_{(124)\lambda\nu} \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(b_1)}_{(124)}, \quad (49)$$

$$U^{\mu}_{b_1 DD} = \tilde{T}^{(2)\mu\lambda}_{(b_1 4)} \tilde{t}^{(2)}_{(\phi 3)\lambda\nu} \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(b_1)}_{(123)} + \tilde{T}^{(2)\mu\lambda}_{(b_1 3)} \tilde{t}^{(2)}_{(\phi 4)\lambda\nu} \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(b_1)}_{(124)}. \quad (50)$$

2.4 $\psi \to \omega K^+K^- \to \pi^+\pi^-\pi^0 K^+K^-$

The formulae for this channel are quite similar to those in the previous subsection for $\psi \to \phi\pi^+\pi^-$. If we number π^0, π^+, π^-, K^+, K^- as 0, 1, 2, 3, 4, then we can get corresponding partial-wave amplitudes by simply replacing $\tilde{t}^{(1)\mu}_{(12)}$ in equations of the previous subsection by ω^{μ} defined as

$$\omega^{\mu} = \epsilon^{\mu}_{\nu\lambda\sigma} p_1^{\nu} p_2^{\lambda} p_0^{\sigma} \Big[B_1(Q_{\omega\rho 0}) f^{(\rho)}_{(12)} B_1(Q_{\rho 12}) + B_1(Q_{\omega\rho 2}) f^{(\rho)}_{(10)} B_1(Q_{\rho 10}) + B_1(Q_{\omega\rho 1}) f^{(\rho)}_{(20)} B_1(Q_{\rho 20}) \Big] \quad (51)$$

and replacing $f^{(\phi)}_{(12)}$ by $f^{(\omega)}_{(012)}$.

2.5 $\psi \to K^+\pi^-K^-\pi^+$

We label K^+, π^-, K^-, π^+ as 1, 2, 3, 4. For ρa_0 and ρa_2 intermediate states, the formulae are the same as for ϕf_0 and ϕf_2 intermediate states with a trivial exchange between pions and kaons. For $KK^{*'} \to (KK^*\pi$ or $K\rho K)$, or $\pi\rho' \to K^*K$ intermediate states, the formulae are the same as for the $\pi\rho' \to \pi\phi\pi$ intermediate state with proper recombination of particles. For $KK_1^* \to (KK^*\pi$ or $K\rho K)$ intermediate states, the formulae are the same as for the $\pi b_1 \to \pi\phi\pi$ intermediate state with proper recombination of particles. So all the formulae given in the subsection on $\phi\pi^+\pi^-$ may be applied here. In addition, there are many more possible intermediate states. We list additional formulae for some obvious large intermediate states. Note that a resonance with negative C-parity decays to $K^*_{j_1}\bar{K}^*_{j_2}$ with a relative minus sign to its charge conjugate state $\bar{K}^*_{j_1}K^*_{j_2}$:

$$\langle K^*K_0^*|01\rangle = \tilde{t}^{(1)\mu}_{(12)} f^{(K^*)}_{(12)} f^{(\bar{K}_0^*)}_{(34)} - \tilde{t}^{(1)\mu}_{(34)} f^{(\bar{K}^*)}_{(34)} f^{(K_0^*)}_{(12)}, \quad (52)$$

$$\langle K^*K_0^*|21\rangle = \tilde{T}^{(2)\mu\nu}_{((12)(34))}\Big[\tilde{t}^{(1)}_{(12)\nu} f^{(K^*)}_{(12)} f^{(K_0^*)}_{(34)} - \tilde{t}^{(1)}_{(34)\nu} f^{(K^*)}_{(34)} f^{(K_0^*)}_{(12)}\Big], \quad (53)$$

$$\langle K^*K_2^*|01\rangle = \tilde{t}^{(2)\mu\nu}_{(34)}\tilde{t}^{(1)}_{(12)\nu} f^{(K^*)}_{(12)} f^{(K_2^*)}_{(34)} - \tilde{t}^{(2)\mu\nu}_{(12)}\tilde{t}^{(1)}_{(34)\nu} f^{(K^*)}_{(34)} f^{(K_2^*)}_{(12)}, \quad (54)$$

$$\langle K^*K_2^*|21\rangle = \tilde{T}^{(2)\mu\alpha}_{((12)(34))}\Big[\tilde{t}^{(2)}_{(34)\alpha\nu}\tilde{t}^{(1)\nu}_{(12)} f^{(K^*)}_{(12)} f^{(K_2^*)}_{(34)} - \tilde{t}^{(2)}_{(12)\alpha\nu}\tilde{t}^{(1)\nu}_{(34)} f^{(K^*)}_{(34)} f^{(K_2^*)}_{(12)}\Big], \quad (55)$$

$$\langle K^*K_2^*|22\rangle = \epsilon^{\mu\alpha\beta\gamma} p_{\psi\alpha}\tilde{T}^{(2)\delta}_{((12)(34))\beta} \cdot\Big\{\Big[\epsilon_{\gamma\lambda\sigma\nu}\tilde{t}^{(2)\lambda}_{(34)\delta} + \epsilon_{\delta\lambda\sigma\nu}\tilde{t}^{(2)\lambda}_{(34)\gamma}\Big] p^\sigma_\psi \tilde{t}^{(1)\nu}_{(12)} f^{(K^*)}_{(12)} f^{(K_2^*)}_{(34)} - \Big[\epsilon_{\gamma\lambda\sigma\nu}\tilde{t}^{(2)\lambda}_{(12)\delta} + \epsilon_{\delta\lambda\sigma\nu}\tilde{t}^{(2)\lambda}_{(12)\gamma}\Big] p^\sigma_\psi \tilde{t}^{(1)\nu}_{(34)} f^{(K^*)}_{(34)} f^{(K_2^*)}_{(12)}\Big\}, \quad (56)$$

$$\langle K^*K_2^*|23\rangle = P^{(3)\mu\alpha\beta\gamma\delta\nu}(p_\psi)\tilde{T}^{(2)}_{((12)(34))\alpha\beta} \cdot\Big[\tilde{t}^{(2)}_{(34)\gamma\delta}\tilde{t}^{(1)}_{(12)\nu} f^{(K^*)}_{(12)} f^{(K_2^*)}_{(34)} - \tilde{t}^{(2)}_{(12)\gamma\delta}\tilde{t}^{(1)}_{(34)\nu} f^{(K^*)}_{(34)} f^{(K_2^*)}_{(12)}\Big], \quad (57)$$

$$\langle K^*K_2^*|43\rangle = \tilde{T}^{(4)\mu\nu\lambda\sigma}_{((12)(34))}\Big[\tilde{t}^{(1)}_{(12)\nu}\tilde{t}^{(2)}_{(34)\lambda\sigma} f^{(K^*)}_{(12)} f^{(K_2^*)}_{(34)} - \tilde{t}^{(1)}_{(34)\nu}\tilde{t}^{(2)}_{(12)\lambda\sigma} f^{(K^*)}_{(34)} f^{(K_2^*)}_{(12)}\Big], \quad (58)$$

$$\langle K_0^*K_0^{*'}|10\rangle = \tilde{T}^{(1)\mu}_{((12)(34))}\Big[f^{K_0^*}_{(12)} f^{K_0^{*'}}_{(34)} + f^{K_0^*}_{(34)} f^{K_0^{*'}}_{(12)}\Big], \quad (59)$$

$$\langle K_0^*K_2^*|12\rangle = \tilde{T}^{(1)}_{((12)(34))\nu}\Big[\tilde{t}^{(2)\mu\nu}_{(34)} f^{(K_0^*)}_{(12)} f^{(K_2^*)}_{(34)} + \tilde{t}^{(2)\mu\nu}_{(12)} f^{(K_0^*)}_{(34)} f^{(K_2^*)}_{(12)}\Big], \quad (60)$$

$$\langle K_0^*K_2^*|32\rangle = \tilde{T}^{(3)\mu\nu\lambda}_{((12)(34))}\Big[\tilde{t}^{(2)}_{(34)\nu\lambda} f^{(K_0^*)}_{(12)} f^{(K_2^*)}_{(34)} + \tilde{t}^{(2)}_{(12)\nu\lambda} f^{(K_0^*)}_{(34)} f^{(K_2^*)}_{(12)}\Big], \quad (61)$$

$$\langle K^*K^{*'}|10\rangle = \tilde{T}^{(1)\mu}_{((12)(34))}\tilde{t}^{(1)\alpha}_{(12)}\tilde{t}^{(1)}_{(34)\alpha} \cdot\Big[f^{K^*}_{(12)} f^{K^{*'}}_{(34)} + f^{K^*}_{(34)} f^{K^{*'}}_{(12)}\Big], \quad (62)$$

$$\langle K^*K^{*'}|11\rangle = \epsilon^{\mu\alpha\beta\gamma} p_{\psi\alpha}\epsilon_{\beta\nu\lambda\sigma} p^\nu_\psi \tilde{t}^{(1)\lambda}_{(12)}\tilde{t}^{(1)\sigma}_{(34)}\tilde{T}^{(1)}_{((12)(34))\gamma} \cdot\Big[f^{K^*}_{(12)} f^{K^{*'}}_{(34)} - f^{K^*}_{(34)} f^{K^{*'}}_{(12)}\Big], \quad (63)$$

$$\langle K^*K^{*'}|12\rangle = P^{(2)\mu\nu\alpha\beta}(p_\psi)\tilde{t}^{(1)}_{(12)\alpha}\tilde{t}^{(1)}_{(34)\beta}\tilde{T}^{(1)}_{((12)(34))\nu} \cdot\Big[f^{K^*}_{(12)} f^{K^{*'}}_{(34)} + f^{K^*}_{(34)} f^{K^{*'}}_{(12)}\Big], \quad (64)$$

$$\langle K^*K^{*'}|32\rangle = \tilde{T}^{(3)\mu\nu\lambda}_{((12)(34))}\tilde{t}^{(1)}_{(12)\nu}\tilde{t}^{(1)}_{(34)\lambda} \cdot\Big[f^{K^*}_{(12)} f^{K^{*'}}_{(34)} + f^{K^*}_{(34)} f^{K^{*'}}_{(12)}\Big]. \quad (65)$$

Smaller contribution from KK_2^* with $K_2^* \to K^*\pi$ or $K\rho$ and some other intermediate states may also need to be considered.

2.6 $\psi \to \phi\pi^+\pi^-\pi^+\pi^- \to K^+K^-\pi^+\pi^-\pi^+\pi^-$

As for the $\psi \to \phi\pi^+\pi^-$ channel, the dominant intermediate states are also ϕf_0 and ϕf_2. The f_0-resonances decay to $\pi^+\pi^-\pi^+\pi^-$ usually through $\sigma\sigma$ and $\rho\rho$; and the f_2-resonances decay to $\pi^+\pi^-\pi^+\pi^-$ usually through $\sigma\sigma$, $\rho\rho$ and $f_2(1270)\sigma$. We assume a similar notation to the $\psi \to \phi\pi^+\pi^-$ case and number the additional $\pi^+\pi^-$ as particle 5, 6. Then the corresponding partial-wave amplitudes involving $f_J \to \sigma\sigma$ are

$$\langle\phi f_0|01\rangle_{(\sigma\sigma)} = \tilde{t}^{(1)\mu}_{(12)} f^{(\phi)}_{(12)} f^{(f_0)}_{(\sigma\sigma)}\Big[f^{(\sigma)}_{(34)} f^{(\sigma)}_{(56)} + f^{(\sigma)}_{(36)} f^{(\sigma)}_{(45)}\Big], \quad (66)$$

$$\langle\phi f_0|21\rangle_{(\sigma\sigma)} = \tilde{T}^{(2)\mu\nu}_{(\phi f_0)}\tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_0)}_{(\sigma\sigma)} \cdot\Big[f^{(\sigma)}_{(34)} f^{(\sigma)}_{(56)} + f^{(\sigma)}_{(36)} f^{(\sigma)}_{(45)}\Big], \quad (67)$$

$$\langle\phi f_2|01\rangle_{(\sigma\sigma)} = T^{(f_2)\mu\nu}_{(\sigma\sigma)}\tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_2)}_{(\sigma\sigma)}, \quad (68)$$

$$\langle\phi f_2|21\rangle_{(\sigma\sigma)} = \tilde{T}^{(2)\mu\alpha}_{(\phi f_2)}\tilde{T}^{(f_2)}_{(\sigma\sigma)\alpha\nu}\tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(f_2)}_{(\sigma\sigma)}, \quad (69)$$

$$\langle\phi f_2|22\rangle_{(\sigma\sigma)} = \epsilon^{\mu\alpha\beta\gamma} p_{\psi\alpha}\tilde{T}^{(2)\delta}_{(\phi f_2)\beta}\Big[\epsilon_{\gamma\lambda\sigma\nu}\tilde{T}^{(f_2)\lambda}_{(\sigma\sigma)\delta} + \epsilon_{\delta\lambda\sigma\nu}\tilde{T}^{(f_2)\lambda}_{(\sigma\sigma)\gamma}\Big] p^\sigma_\psi \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(f_2)}_{(\sigma\sigma)}, \quad (70)$$

$$\langle\phi f_2|23\rangle_{(\sigma\sigma)} = P^{(3)\mu\alpha\beta\gamma\delta\nu}(p_\psi)\tilde{T}^{(2)}_{(\phi f_2)\alpha\beta} \cdot\tilde{T}^{(f_2)}_{(\sigma\sigma)\gamma\delta}\tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_2)}_{(\sigma\sigma)}, \quad (71)$$

$$\langle\phi f_2|43\rangle_{(\sigma\sigma)} = \tilde{T}^{(4)\mu\nu\lambda\sigma}_{(\phi f_2)}\tilde{t}^{(1)}_{(12)\nu} T^{(f_2)}_{(\sigma\sigma)\lambda\sigma} f^{(\phi)}_{(12)} f^{(f_2)}_{(\sigma\sigma)} \quad (72)$$

with

$$T^{(f_2)\mu\nu}_{(\sigma\sigma)} = \tilde{t}^{(2)\mu\nu}_{(\sigma_{34}\sigma_{56})} f^{(\sigma)}_{(34)} f^{(\sigma)}_{(56)} + \tilde{t}^{(2)\mu\nu}_{(\sigma_{36}\sigma_{45})} f^{(\sigma)}_{(36)} f^{(\sigma)}_{(45)}. \quad (73)$$

For $f_J \to \rho\rho$, if we limit $\rho\rho$ to a relative $l=0$ state, then the corresponding partial-wave amplitudes are

$$\langle\phi f_0|01\rangle_{(\rho\rho)} = \tilde{t}^{(1)\mu}_{(12)} f^{(\phi)}_{(12)} f^{(f_0)}_{(\rho\rho)} \Big[f^{(\rho)}_{(34)} f^{(\rho)}_{(56)} \tilde{t}^{(1)\alpha\beta}_{(34)} \tilde{t}^{(1)}_{(56)\alpha\beta} + f^{(\rho)}_{(36)} f^{(\rho)}_{(45)} \tilde{t}^{(1)\alpha\beta}_{(36)} \tilde{t}^{(1)}_{(45)\alpha\beta} \Big], \quad (74)$$

$$\langle\phi f_0|21\rangle_{(\rho\rho)} = \tilde{T}^{(2)\mu\nu}_{(\phi f_0)} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_0)}_{(\rho\rho)} \Big[f^{(\rho)}_{(34)} f^{(\rho)}_{(56)} \tilde{t}^{(1)\alpha\beta}_{(34)} \tilde{t}^{(1)}_{(56)\alpha\beta} + f^{(\rho)}_{(36)} f^{(\rho)}_{(45)} \tilde{t}^{(1)\alpha\beta}_{(36)} \tilde{t}^{(1)}_{(45)\alpha\beta} \Big], \quad (75)$$

$$\langle\phi f_2|01\rangle_{(\rho\rho)} = T^{(f_2)\mu\nu}_{(\rho\rho)} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_2)}_{(\rho\rho)}, \quad (76)$$

$$\langle\phi f_2|21\rangle_{(\rho\rho)} = \tilde{T}^{(2)\mu\alpha}_{(\phi f_2)} \tilde{T}^{(f_2)}_{(\rho\rho)\alpha\nu} \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(f_2)}_{(\rho\rho)}, \quad (77)$$

$$\langle\phi f_2|22\rangle_{(\rho\rho)} = \epsilon^{\mu\alpha\beta\gamma} p_{\psi\alpha} \tilde{T}^{(2)\delta}_{(\phi f_2)\beta} \Big[\epsilon_{\gamma\lambda\sigma\nu} \tilde{T}^{(f_2)\lambda}_{(\rho\rho)\delta} + \epsilon_{\delta\lambda\sigma\nu} \tilde{T}^{(f_2)\lambda}_{(\rho\rho)\gamma} \Big] p^{\sigma}_{\psi} \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(f_2)}_{(\rho\rho)}, \quad (78)$$

$$\langle\phi f_2|23\rangle_{(\rho\rho)} = P^{(3)\mu\alpha\beta\gamma\delta\nu}(p_\psi) \tilde{T}^{(2)}_{(\phi f_2)\alpha\beta} \cdot \tilde{T}^{(f_2)}_{(\rho\rho)\gamma\delta} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_2)}_{(\rho\rho)}, \quad (79)$$

$$\langle\phi f_2|43\rangle_{(\rho\rho)} = \tilde{T}^{(4)\mu\nu\lambda\sigma}_{(\phi f_2)} \tilde{t}^{(1)}_{(12)\nu} T^{(f_2)}_{(\rho\rho)\lambda\sigma} f^{(\phi)}_{(12)} f^{(f_2)}_{(\rho\rho)}, \quad (80)$$

where

$$T^{(f_2)\mu\nu}_{(\rho\rho)} = P^{(2)\mu\nu\alpha\beta}(p_{f_2}) \Big[\tilde{t}^{(1)}_{(34)\alpha} \tilde{t}^{(1)}_{(56)\beta} f^{(\rho)}_{(34)} f^{(\rho)}_{(56)} + \tilde{t}^{(1)}_{(36)\alpha} \tilde{t}^{(1)}_{(45)\beta} f^{(\rho)}_{(36)} f^{(\rho)}_{(45)} \Big]. \quad (81)$$

For $f_2 \to f_2(1270)\sigma$, if we also limit $f_2(1270)\sigma$ to the $l=0$ state, then we have the corresponding partial-wave amplitudes:

$$\langle\phi f_2|01\rangle_{(f_2\sigma)} = T^{(f_2)\mu\nu}_{(f_2\sigma)} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_2)}_{(f_2\sigma)}, \quad (82)$$

$$\langle\phi f_2|21\rangle_{(f_2\sigma)} = \tilde{T}^{(2)\mu\alpha}_{(\phi f_2)} \tilde{T}^{(f_2)}_{(f_2\sigma)\alpha\nu} \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(f_2)}_{(f_2\sigma)}, \quad (83)$$

$$\langle\phi f_2|22\rangle_{(f_2\sigma)} = \epsilon^{\mu\alpha\beta\gamma} p_{\psi\alpha} \tilde{T}^{(2)\delta}_{(\phi f_2)\beta} \Big[\epsilon_{\gamma\lambda\sigma\nu} \tilde{T}^{(f_2)\lambda}_{(f_2\sigma)\delta} + \epsilon_{\delta\lambda\sigma\nu} \tilde{T}^{(f_2)\lambda}_{(f_2\sigma)\gamma} \Big] p^{\sigma}_{\psi} \tilde{t}^{(1)\nu}_{(12)} f^{(\phi)}_{(12)} f^{(f_2)}_{(f_2\sigma)}, \quad (84)$$

$$\langle\phi f_2|23\rangle_{(f_2\sigma)} = P^{(3)\mu\alpha\beta\gamma\delta\nu}(p_\psi) \tilde{T}^{(2)}_{(\phi f_2)\alpha\beta} \cdot \tilde{T}^{(f_2)}_{(f_2\sigma)\gamma\delta} \tilde{t}^{(1)}_{(12)\nu} f^{(\phi)}_{(12)} f^{(f_2)}_{(f_2\sigma)}, \quad (85)$$

$$\langle\phi f_2|43\rangle_{(f_2\sigma)} = \tilde{T}^{(4)\mu\nu\lambda\sigma}_{(\phi f_2)} \tilde{t}^{(1)}_{(12)\nu} T^{(f_2)}_{(f_2\sigma)\lambda\sigma} f^{(\phi)}_{(12)} f^{(f_2)}_{(f_2\sigma)} \quad (86)$$

with

$$T^{(f_2)\mu\nu}_{(f_2\sigma)} = P^{(2)\mu\nu\alpha\beta}(p_{f_2}) \cdot \Big[\tilde{t}^{(2)}_{(34)\alpha\beta} f^{(f_2)}_{(34)} f^{(\sigma)}_{(56)} + \tilde{t}^{(2)}_{(56)\alpha\beta} f^{(f_2)}_{(56)} f^{(\sigma)}_{(34)} + \tilde{t}^{(2)}_{(36)\alpha\beta} f^{(f_2)}_{(36)} f^{(\sigma)}_{(45)} + \tilde{t}^{(2)}_{(45)\alpha\beta} f^{(f_2)}_{(45)} f^{(\sigma)}_{(36)} \Big]. \quad (87)$$

Unlike the $\psi \to \phi\pi^+\pi^-$ channel, for $\psi \to \phi\pi^+\pi^-\pi^+\pi^-$ it is possible to go through 0^{-+} resonances (η^*) decaying to $\pi^+\pi^-\pi^+\pi^-$ through $\rho\rho$. The corresponding partial wave is

$$U^{\mu}_{\eta^*} = \epsilon^{\mu\nu\alpha\beta} \tilde{t}^{(1)}_{(12)\nu} \tilde{T}^{(1)}_{(\phi\eta^*)\alpha} p_{\psi\beta} \epsilon_{\tau\sigma\gamma\eta} p^{\tau}_3 p^{\sigma}_4 p^{\gamma}_5 p^{\eta}_6 \cdot \Big[f^{(\rho)}_{(34)} f^{(\rho)}_{(56)} B_1(Q_{\rho 34}) B_1(Q_{\rho 56}) B_1(Q_{\eta^*\rho_{34}\rho_{56}}) - f^{(\rho)}_{(36)} f^{(\rho)}_{(45)} B_1(Q_{\rho 36}) B_1(Q_{\rho 45}) B_1(Q_{\eta^*\rho_{36}\rho_{45}}) \Big]. \quad (88)$$

Besides partial-wave amplitudes given above, for $\pi^+\pi^-\pi^+\pi^-$ final states, there are many other possible intermediate states, such as $a_2\pi$, $a_1\pi$, $\pi(1300)\pi$ etc. Before performing the partial-wave analysis, one should check various invariant mass spectra to see what resonances are present in the data and add the corresponding partial-wave amplitudes.

3 Formalism for ψ radiative decay to mesons

We denote the ψ polarization four-vector by $\psi_\mu(m_1)$ and the polarization vector of the photon by $e_\nu(m_2)$. Then the general form for the decay amplitude is

$$A = \psi_\mu(m_1) e^*_\nu(m_2) A^{\mu\nu} = \psi_\mu(m_1) e^*_\nu(m_2) \sum_i \Lambda_i U^{\mu\nu}_i. \quad (89)$$

For the photon polarization four-vector e_ν with photon momentum q, there is the usual Lorentz orthogonality condition $e_\nu q^\nu = 0$. This is the same as for a massive vector meson. However, for the photon, there is an additional gauge invariance condition. Here we assume the Coulomb gauge in the ψ rest system, *i.e.*, $e_\nu p^\nu_\psi = 0$. Then we have [13]

$$\sum_m e^*_\mu(m) e_\nu(m) = -g_{\mu\nu} + \frac{q_\mu K_\nu + K_\mu q_\nu}{q\cdot K} - \frac{K\cdot K}{(q\cdot K)^2} q_\mu q_\nu \equiv -g^{(\perp\perp)}_{\mu\nu} \quad (90)$$

with $K = p_\psi - q$ and $e_\nu K^\nu = 0$. The radiative decay cross-section is

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_n} &= \frac{1}{2} \sum_{m_1=1}^{2} \sum_{m_2=1}^{2} \psi_\mu(m_1) e^*_\nu(m_2) A^{\mu\nu} \psi^*_{\mu'}(m_1) e_{\nu'}(m_2) A^{*\mu'\nu'} \\ &= -\frac{1}{2} \sum_{m_1=1}^{2} \psi_\mu(m_1) \psi^*_{\mu'}(m_1) g^{(\perp\perp)}_{\nu\nu'} A^{\mu\nu} A^{*\mu'\nu'} \\ &= -\frac{1}{2} \sum_{\mu=1}^{2} A_{\mu\nu} g^{(\perp\perp)}_{\nu\nu'} A^{*\mu\nu'} \\ &= -\frac{1}{2} \sum_{i,j} \Lambda_i \Lambda^*_j \sum_{\mu=1}^{2} U^{\mu\nu}_i g^{(\perp\perp)}_{\nu\nu'} U^{*\mu\nu'}_j \equiv \sum_{i,j} P_{ij} \cdot F_{ij}, \end{aligned} \quad (91)$$

where

$$P_{ij} = P^*_{ji} = \Lambda_i \Lambda^*_j, \quad (92)$$

$$F_{ij} = F^*_{ji} = -\frac{1}{2} \sum_{\mu=1}^{2} U^{\mu\nu}_i g^{(\perp\perp)}_{\nu\nu'} U^{*\mu\nu'}_j. \quad (93)$$

Due to the special properties (massless and gauge invariance) of the photon, the number of independent partial-wave amplitudes for a ψ radiative decay is smaller than for the corresponding decay to a massive vector meson. For example, for $\psi \to \phi f_0$, there are two independent partial-wave amplitudes with orbital angular momentum $L = 0$ and 2, respectively, which give different angular distributions; but for $\psi \to \gamma f_0$, with the gauge invariance condition, the two amplitudes will give the same angular distribution. So, for the ψ radiative decay, the L-S scheme is not useful any more for choosing independent amplitudes. One may simply use momenta of the particles to construct covariant tensor amplitudes; it is sufficient to check the helicity amplitudes to make sure there is the right number of independent amplitudes. From the helicity formalism, it is easy to show that there is one independent amplitude for ψ radiative decay to a spin-0 meson, two independent amplitudes for ψ radiative decay to a spin-1 meson, and three independent amplitudes for ψ radiative decay to a meson with spin larger than 1.

3.1 ψ radiative decay to two pseudoscalar mesons

We denote the two pseudoscalar mesons as π^+ and π^-. For the decay vertex $\psi \to \gamma f_J$, there are two independent momenta which we choose to be p_ψ and the momentum of the photon q. We use these two momenta and the spin wave functions of the three particles to construct the covariant tensor amplitudes.

For $\psi \to \gamma f_0$, the e_μ can only contract with ψ^μ since $e_\mu p_\psi^\mu = e_\mu q^\mu = 0$; hence there is only one independent amplitude:

$$U^{\mu\nu}_{\gamma f_0} = g^{\mu\nu} f^{(f_0)}. \tag{94}$$

For $\psi \to \gamma f_2$ or $\psi \to \gamma f_4$, e_μ may contract with ψ^μ or with the spin wave function of f_J. Then ψ^μ may contract with e_μ, or q_μ, or the spin wave function of f_J; this gives three independent covariant tensor amplitudes for each f_J:

$$U^{\mu\nu}_{(\gamma f_2)1} = \tilde{t}^{(f_2)\mu\nu} f^{(f_2)}, \tag{95}$$

$$U^{\mu\nu}_{(\gamma f_2)2} = g^{\mu\nu} p_\psi^\alpha p_\psi^\beta \tilde{t}^{(f_2)}_{\alpha\beta} B_2(Q_{\Psi\gamma f_2}) f^{(f_2)}, \tag{96}$$

$$U^{\mu\nu}_{(\gamma f_2)3} = q^\mu \tilde{t}^{(f_2)\nu}_\alpha p_\psi^\alpha B_2(Q_{\psi\gamma f_2}) f^{(f_2)}, \tag{97}$$

$$U^{\mu\nu}_{(\gamma f_4)1} = \tilde{t}^{(f_4)\mu\nu}_{\alpha\beta} p_\psi^\alpha p_\psi^\beta B_2(Q_{\Psi\gamma f_4}) f^{(f_4)}, \tag{98}$$

$$U^{\mu\nu}_{(\gamma f_4)2} = g^{\mu\nu} \tilde{t}^{(f_4)}_{\alpha\beta\gamma\delta} p_\psi^\alpha p_\psi^\beta p_\psi^\gamma p_\psi^\delta B_4(Q_{\psi\gamma f_4}) f^{(f_4)}, \tag{99}$$

$$U^{\mu\nu}_{(\gamma f_4)3} = q^\mu \tilde{t}^{(f_4)\nu}_{\alpha\beta\gamma} p_\psi^\alpha p_\psi^\beta p_\psi^\gamma B_4(Q_{\Psi\gamma f_4}) f^{(f_4)}, \tag{100}$$

where

$$\begin{aligned}\tilde{t}^{(f_J)}_{\mu_1\cdots\mu_J} &= \sum_m \phi_{\mu_1\cdots\mu_J}(p_{f_J}, m) \\ &\quad \cdot \phi_{\mu'_1\cdots\mu'_J}(p_{f_J}, m) r_\pi^{\mu'_1} \cdots r_\pi^{\mu'_J} B_J(Q_{f_J\pi\pi}) \\ &= P_{\mu_1\cdots\mu_J\mu'_1\cdots\mu'_J}(p_{f_J}) r_\pi^{\mu'_1} \cdots r_\pi^{\mu'_J} B_J(Q_{f_J\pi\pi})\end{aligned} \tag{101}$$

with $J = 0$, 2; here r_π represents the relative momentum between two pseudoscalar mesons.

We use p_ψ instead of q to contract with $\tilde{t}^{(f_J)}$ because $q\tilde{t}^{(f_J)} = p_\psi \tilde{t}^{(f_J)}$ and p_ψ has only a time component in the ψ rest system. This makes the calculation simpler.

3.2 $\psi \to \gamma\eta\pi^+\pi^-$

This is a three-step process: $\psi \to \gamma X$ with $X \to yz$ and $y \to \pi\pi$ or $y \to \eta\pi$. The amplitudes $U^i_{\mu\nu}$ are listed using the notation

$$\langle \gamma J^{PC} | (yz) i \rangle , \tag{102}$$

where J, P, C are the intrinsic spin, parity and C-parity of the X-particle, respectively. We denote π^+, π^-, η as 1, 2, 3, respectively. The possible J^{PC} for X are 0^{-+}, 1^{++}, 1^{-+}, 2^{++}, 2^{-+}, 3^{++}, 3^{-+}, etc. For invariant mass below 2 GeV, we consider J up to 2. For $\psi \to \gamma X$, we choose two independent momenta p_ψ for ψ and q for the photon to be contracted with spin wave functions.

For the $\psi \to \gamma 0^{-+}$ vertex, there is only one independent coupling, $\epsilon_{\mu\nu\lambda\sigma}\psi^\mu e^\nu q^\lambda p_\psi^\sigma$. With various possible yz states, we have $U^i_{\mu\nu}$ for $\psi \to \gamma 0^- \to \eta\pi^+\pi^-$ as follows:

$$\langle \gamma 0^{-+} | (f_0\eta) 1 \rangle = S_{\mu\nu} B_1(Q_{\psi\gamma X}) f^{(f_0)}_{(12)}, \tag{103}$$

$$\langle \gamma 0^{-+} | (a_0\pi) 1 \rangle = S_{\mu\nu} B_1(Q_{\psi\gamma X}) (f^{(a_0)}_{(13)} + f^{(a_0)}_{(23)}), \tag{104}$$

$$\langle \gamma 0^{-+} | (f_2\eta) 1 \rangle = S_{\mu\nu} B_1(Q_{\psi\gamma X}) f^{(f_2)}_{(12)} \tilde{t}^{(2)}_{(f_2\eta)\gamma\delta} \tilde{t}^{(2)\gamma\delta}_{(12)}, \tag{105}$$

$$\begin{aligned}\langle \gamma 0^{-+} | (a_2\pi) 1 \rangle = S_{\mu\nu} B_1(Q_{\psi\gamma X}) \Big\{ & f^{(a_2)}_{(13)} \tilde{t}^{(2)}_{(a_2 2)\gamma\delta} \tilde{t}^{(2)\gamma\delta}_{(13)} \\ & + f^{(a_2)}_{(23)} \tilde{t}^{(2)}_{(a_2 1)\gamma\delta} \tilde{t}^{(2)\gamma\delta}_{(23)} \Big\}\end{aligned} \tag{106}$$

with $S_{\mu\nu}$ defined as

$$S_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} p_\psi^\alpha q^\beta. \tag{107}$$

For the $\psi \to \gamma 1^{++}$ vertex, there are two independent couplings for each yz:

$$\langle \gamma 1^{++} | (f_0\eta) 1 \rangle = \epsilon_{\mu\nu\alpha\beta} p_\psi^\alpha \tilde{t}^{(1)\beta}_{(\eta f_0)} f^{(f_0)}_{(12)}, \tag{108}$$

$$\langle \gamma 1^{++} | (a_0\pi) 1 \rangle = \epsilon_{\mu\nu\alpha\beta} p_\psi^\alpha (\tilde{t}^{(1)\beta}_{(a_0 1)} f^{(a_0)}_{(23)} + \tilde{t}^{(1)\beta}_{(a_0 2)} f^{(a_0)}_{(13)}), \tag{109}$$

$$\langle \gamma 1^{++} | (f_0\eta) 2 \rangle = q_\mu S_{\nu\beta} \tilde{t}^{(1)\beta}_{(\eta f_0)} B_2(Q_{\psi\gamma X}) f^{(f_0)}_{(12)}, \tag{110}$$

$$\begin{aligned}\langle \gamma 1^{++} | (a_0\pi) 2 \rangle = q_\mu S_{\nu\beta} B_2(Q_{\psi\gamma X}) \\ \cdot \left[t^{(1)\beta}_{(a_0 1)} f^{(a_0)}_{(23)} + t^{(1)\beta}_{(a_0 2)} f^{(a_0)}_{(13)} \right],\end{aligned} \tag{111}$$

$$\langle \gamma 1^{++} | (f_2\eta) 1 \rangle = \epsilon_{\mu\nu\alpha\beta} p_\psi^\alpha \tilde{g}^{\beta\gamma}_X \tilde{t}^{(2)}_{(12)\gamma\delta} \tilde{t}^{(1)\delta}_{(\eta f_2)} f^{(f_2)}_{(12)}, \tag{112}$$

$$\begin{aligned}\langle \gamma 1^{++} | (a_2\pi) 1 \rangle = \epsilon_{\mu\nu\alpha\beta} p_\psi^\alpha \tilde{g}^{\beta\gamma}_X \Big[& \tilde{t}^{(2)}_{(13)\gamma\delta} \tilde{t}^{(1)\delta}_{(a_2 2)} f^{(a_2)}_{(13)} \\ & + \tilde{t}^{(2)}_{(23)\gamma\delta} \tilde{t}^{(1)\delta}_{(a_2 1)} f^{(a_2)}_{(23)} \Big]\end{aligned} \tag{113}$$

$$\langle \gamma 1^{++} | (f_2\eta) 2 \rangle = q_\mu S_{\nu\beta} \tilde{g}^{\beta\beta'}_X \tilde{t}_{(12)\beta'\alpha'} \tilde{t}^{(1)\alpha'}_{(\eta f_2)} B_2(Q_{\psi\gamma X}) f^{(f_2)}_{(12)}, \tag{114}$$

$$\begin{aligned}\langle \gamma 1^{++} | (a_2\pi) 2 \rangle = q_\mu S_{\nu\beta} \tilde{g}^{\beta\beta'}_X B_2(Q_{\psi\gamma X}) \\ \cdot \left[\tilde{t}^{(2)}_{(13)\beta'\alpha'} \tilde{t}^{(1)\alpha'}_{(a_2 2)} f^{(a_2)}_{(13)} + \tilde{t}^{(2)}_{(23)\beta'\alpha'} \tilde{t}^{(1)\alpha'}_{(a_2 1)} f^{(a_2)}_{(23)} \right],\end{aligned} \tag{115}$$

where $\tilde{g}_X^{\alpha\beta} = g^{\alpha\beta} - \frac{p_X^\alpha p_X^\beta}{p_X^2}$. For 1^{++} decaying to $f_2\eta$ and $a_2\pi$, the orbital angular momentum l could be 1 and 3; but we ignore the $l=3$ contribution because of the strong centrifugal barrier.

For $\psi \to \gamma 1^{-+}$, the exotic 1^{-+} meson cannot decay into $f_0\eta$ and $a_0\pi$. We have four $U^i_{\mu\nu}$ amplitudes here:

$$\langle\gamma 1^{-+}|(f_2\eta)1\rangle = g_{\mu\nu} S_{\gamma\delta} \tilde{t}^{(2)\gamma\sigma}_{(\eta f_2)} \tilde{t}^{(2)\delta}_{(12)\sigma} f^{(f_2)}_{(12)} B_1(Q_{\psi\gamma X}), \quad (116)$$

$$\langle\gamma 1^{-+}|(a_2\pi)1\rangle = g_{\mu\nu} S_{\gamma\delta} \Big[\tilde{t}^{(2)\gamma\sigma}_{(a_2 2)} \tilde{t}^{(2)\delta}_{(13)\sigma} f^{(a_2)}_{(13)} + \tilde{t}^{(2)\gamma\sigma}_{(a_2 1)} \tilde{t}^{(2)\delta}_{(23)\sigma} f^{(a_2)}_{(23)}\Big] B_1(Q_{\psi\gamma X}), \quad (117)$$

$$\langle\gamma 1^{-+}|(f_2\eta)2\rangle = q_\mu \epsilon_{\nu\beta\gamma\delta} K^\beta \tilde{t}^{(2)\gamma\sigma}_{(\eta f_2)} \tilde{t}^{(2)\delta}_{(12)\sigma} f^{(f_2)}_{(12)} B_1(Q_{\psi\gamma X}), \quad (118)$$

$$\langle\gamma 1^{-+}|(a_2\pi)2\rangle = q_\mu \epsilon_{\nu\beta\gamma\delta} K^\beta \Big[\tilde{t}^{(2)\gamma\sigma}_{(a_2 2)} \tilde{t}^{(2)\delta}_{(13)\sigma} f^{(a_2)}_{(13)} + \tilde{t}^{(2)\gamma\sigma}_{(a_2 1)} \tilde{t}^{(2)\delta}_{(23)\sigma} f^{(a_2)}_{(23)}\Big] B_1(Q_{\psi\gamma X}). \quad (119)$$

For $\psi \to \gamma 2^{++}$, there are three independent couplings and two possible yz states, $f_2\eta$ and $a_2\pi$:

$$\langle\gamma 2^{++}|(f_2\eta)1\rangle = P^{(2)}_{\mu\nu\alpha\lambda}(K) \epsilon^{\alpha\beta\gamma\delta} K_\beta \tilde{t}^{(1)}_{(\eta f_2)\gamma} \tilde{t}^{(2)\lambda}_{(12)\delta} f^{(f_2)}_{(12)} \quad (120)$$

$$\langle\gamma 2^{++}|(a_2\pi)1\rangle = P^{(2)}_{\mu\nu\alpha\lambda}(K) \epsilon^{\alpha\beta\gamma\delta} K_\beta \cdot\Big[\tilde{t}^{(1)}_{(a_2 1)\gamma} \tilde{t}^{(2)\lambda}_{(23)\delta} f^{(a_2)}_{(23)} + \tilde{t}^{(1)}_{(a_2 2)\gamma} \tilde{t}^{(2)\lambda}_{(13)\delta} f^{(a_2)}_{(13)}\Big], \quad (121)$$

$$\langle\gamma 2^{++}|(f_2\eta)2\rangle = g_{\mu\nu} p^\lambda_\psi p^\sigma_\psi P^{(2)}_{\lambda\sigma\alpha\beta}(K) B_2(Q_{\psi\gamma X}) \cdot\epsilon^\alpha_{\gamma\delta\alpha'} K^\gamma \tilde{t}^{(1)\delta}_{(\eta f_2)} \tilde{t}^{(2)\alpha'\beta}_{(12)} f^{(f_2)}_{(12)}, \quad (122)$$

$$\langle\gamma 2^{++}|(a_2\pi)2\rangle = g_{\mu\nu} p^\lambda_\psi p^\sigma_\psi P^{(2)}_{\lambda\sigma\alpha\beta}(K) \epsilon^\alpha_{\gamma\delta\alpha'} K^\gamma \cdot\Big[\tilde{t}^{(1)}_{(a_2 1)\delta} \tilde{t}^{(2)\alpha'\lambda}_{(23)} f^{(a_2)}_{(23)} + \tilde{t}^{(1)}_{(a_2 2)\delta} \tilde{t}^{(2)\alpha'\lambda}_{(13)} f^{(a_2)}_{(13)}\Big] B_2(Q_{\psi\gamma X}), \quad (123)$$

$$\langle\gamma 2^{++}|(f_2\eta)3\rangle = q_\mu p^\lambda_\psi P^{(2)}_{\nu\lambda\alpha\beta}(K) B_2(Q_{\Psi\gamma X}) \cdot\epsilon^\alpha_{\gamma\delta\alpha'} K^\gamma \tilde{t}^{(1)\delta}_{(\eta f_2)} \tilde{t}^{(2)\alpha'\beta}_{(12)} f^{(f_2)}_{(12)}, \quad (124)$$

$$\langle\gamma 2^{++}|(a_2\pi)3\rangle = q_\mu p^\lambda_\psi P^{(2)}_{\nu\lambda\alpha\beta}(K) \epsilon^\alpha_{\gamma\delta\alpha'} K^\gamma \cdot\Big[\tilde{t}^{(1)}_{(a_2 1)\delta} \tilde{t}^{(2)\alpha'\beta}_{(23)} f^{(a_2)}_{(23)} + \tilde{t}^{(1)}_{(a_2 2)\delta} \tilde{t}^{(2)\alpha'\beta}_{(13)} f^{(a_2)}_{(13)}\Big] B_2(Q_{\psi\gamma X}). \quad (125)$$

For $\psi \to \gamma 2^{-+}$, we have

$$\langle\gamma 2^{-+}|(f_0\eta)1\rangle = \epsilon_{\mu\nu\alpha\beta} p^\alpha_\psi \tilde{t}^{(2)\beta\gamma}_{(f_0\eta)} q_\gamma f^{(f_0)}_{(12)} B_1(Q_{\psi\gamma X}), \quad (126)$$

$$\langle\gamma 2^{-+}|(a_0\pi)1\rangle = \epsilon_{\mu\nu\alpha\beta} p^\alpha_\psi \cdot\Big\{\tilde{t}^{(2)\beta\gamma}_{(a_0 1)} f^{(a_0)}_{(23)} + \tilde{t}^{(2)\beta\gamma}_{(a_0 2)} f^{(a_0)}_{(13)}\Big\} q_\gamma B_1(Q_{\psi\gamma X}), \quad (127)$$

$$\langle\gamma 2^{-+}|(f_0\eta)2\rangle = S_{\mu\nu} p_{\psi\gamma} p_{\psi\delta} \tilde{t}^{(2)\gamma\delta}_{(f_0\eta)} f^{(f_0)}_{(12)} B_3(Q_{\psi\gamma X}), \quad (128)$$

$$\langle\gamma 2^{-+}|(a_0\pi)2\rangle = S_{\mu\nu} p_{\psi\gamma} p_{\psi\delta} \cdot\Big\{\tilde{t}^{(2)\gamma\delta}_{(a_0 1)} f^{(a_0)}_{(23)} + \tilde{t}^{(2)\gamma\delta}_{(a_0 2)} f^{(a_0)}_{(13)}\Big\} B_3(Q_{\psi\gamma X}), \quad (129)$$

$$\langle\gamma 2^{-+}|(f_0\eta)3\rangle = q_\mu S_{\nu\gamma} \tilde{t}^{(2)\gamma\delta}_{(f_0\eta)} p_{\psi\delta} f^{(f_0)}_{(12)} B_3(Q_{\psi\gamma X}), \quad (130)$$

$$\langle\gamma 2^{-+}|(a_0\pi)3\rangle = q_\mu S_{\nu\gamma} p_{\psi\delta} \cdot\Big\{\tilde{t}^{(2)\gamma\delta}_{(a_0 1)} f^{(a_0)}_{(23)} + \tilde{t}^{(2)\gamma\delta}_{(a_0 2)} f^{(a_0)}_{(13)}\Big\} B_3(Q_{\Psi\gamma X}), \quad (131)$$

$$\langle\gamma 2^{-+}|(f_2\eta)1\rangle = \epsilon_{\mu\nu\alpha\beta} p^\alpha_\psi q_\gamma P^{(2)\beta\gamma\beta'\gamma'}(K) \cdot \tilde{t}^{(2)}_{(12)\beta'\gamma'} f^{(f_2)}_{(12)} B_1(Q_{\psi\gamma X}) \quad (132)$$

$$\langle\gamma 2^{-+}|(a_2\pi)1\rangle = \epsilon_{\mu\nu\alpha\beta} p^\alpha_\psi P^{(2)\beta\gamma\beta'\gamma'}(K) q_\gamma \cdot B_1(Q_{\psi\gamma X}) [\tilde{t}^{(2)}_{(23)\beta'\gamma'} f^{(a_2)}_{(23)} + \tilde{t}^{(2)}_{(13)\beta'\gamma'} f^{(a_2)}_{(13)}], \quad (133)$$

$$\langle\gamma 2^{-+}|(f_2\eta)2\rangle = S_{\mu\nu} p^\gamma_\psi p^\delta_\psi P^{(2)}_{\gamma\delta\gamma'\delta'}(K) \cdot \tilde{t}^{(2)\gamma'\delta'}_{(12)} f^{(f_2)}_{(12)} B_3(Q_{\psi\gamma X}), \quad (134)$$

$$\langle\gamma 2^{-+}|(a_2\pi)2\rangle = S_{\mu\nu} p^\gamma_\psi p^\delta_\psi P^{(2)}_{\gamma\delta\gamma'\delta'}(K) \cdot B_3(Q_{\psi\gamma X}) [\tilde{t}^{(2)\gamma'\delta'}_{(23)} f^{(a_2)}_{(23)} + \tilde{t}^{(2)\gamma'\delta'}_{(13)} f^{(a_2)}_{(13)}], \quad (135)$$

$$\langle\gamma 2^{-+}|(f_2\eta)3\rangle = q_\mu S_{\nu\gamma} p_{\psi\delta} P^{(2)\gamma\delta\gamma'\delta'}(K) \cdot \tilde{t}^{(2)}_{(12)\gamma'\delta'} f^{(f_2)}_{(12)} B_3(Q_{\Psi\gamma X}), \quad (136)$$

$$\langle\gamma 2^{-+}|(a_2\pi)3\rangle = q_\mu S_{\nu\gamma} p_{\psi\delta} P^{(2)\gamma\delta\gamma'\delta'}(K) \cdot B_3(Q_{\Psi\gamma X}) [\tilde{t}^{(2)}_{(23)\gamma'\delta'} f^{(a_2)}_{(23)} + \tilde{t}^{(2)}_{(13)\gamma'\delta'} f^{(a_2)}_{(13)}] \quad (137)$$

with $S_{\mu\nu}$ defined as in eq. (107).

3.3 $\psi \to \gamma K\bar{K}\pi$

Possible intermediate channels for this process are K^*K, K^*_0K, K^*_2K, $a_0\pi$, $a_2\pi$. The formulae for K^*_0K, K^*_2K, $a_0\pi$, $a_2\pi$ intermediate states to the $K\bar{K}\pi$ final state are the same as for the $a_0\pi$, $a_2\pi$, $f_0\eta$, $f_2\eta$ intermediate states given in the previous subsection for the $\pi^+\pi^-\eta$ final state. So, here we only give partial-wave amplitudes $U^i_{\mu\nu}$ with K^*K intermediate states. We denote K, $\bar{K}$, π as particle 1, 2, 3:

$$\langle\gamma 0^{-+}|(K^*K)1\rangle = S_{\mu\nu} B_1(Q_{\psi\gamma X}) \cdot\Big[\tilde{t}^{(1)}_{(K^*\bar{K})\lambda} \tilde{t}^{(1)\lambda}_{(13)} f^{(K^*)}_{(13)} + \tilde{t}^{(2)}_{(K^*\bar{K})\lambda} \tilde{t}^{(1)\lambda}_{(23)} f^{(K^*)}_{(23)}\Big], \quad (138)$$

$$\langle\gamma 1^{++}|(K^*K)1\rangle = \epsilon_{\mu\nu\alpha\beta} p^\alpha_\psi \Big[\tilde{t}^{(1)\beta}_{(23)} f^{(K^*)}_{(23)} + \tilde{t}^{(1)\beta}_{(13)} f^{(K^*)}_{(13)}\Big], \quad (139)$$

$$\langle\gamma 1^{++}|(K^*K)2\rangle = q_\mu S_{\nu\beta}B_2(Q_{\psi\gamma X}) \cdot\left[\tilde{t}^{(1)\beta}_{(23)}f^{(K^*)}_{(23)} + \tilde{t}^{(1)\beta}_{(13)}f^{(K^*)}_{(13)}\right], \quad (140)$$

$$\langle\gamma 1^{-+}|(K^*K)1\rangle = g_{\mu\nu}S_{\gamma\delta}\left[\tilde{t}^{(1)\gamma}_{(K^*\bar{K})}\tilde{t}^{(1)\delta}_{(13)}f^{(K^*)}_{(13)} + \tilde{t}^{(1)\gamma}_{(\bar{K}^*K)}\tilde{t}^{(1)\delta}_{(23)}f^{(K^*)}_{(23)}\right]B_1(Q_{\psi\gamma X}), \quad (141)$$

$$\langle\gamma 1^{-+}|(K^*K)2\rangle = q_\mu\epsilon_{\nu\beta\gamma\delta}K^\beta\left[\tilde{t}^{(1)\gamma}_{(K^*\bar{K})}\tilde{t}^{(1)\delta}_{(13)}f^{(K^*)}_{(13)} + \tilde{t}^{(1)\gamma}_{(\bar{K}^*K)}\tilde{t}^{(1)\delta}_{(23)}f^{(K^*)}_{(23)}\right]B_1(Q_{\psi\gamma X}). \quad (142)$$

3.4 $\psi \to \gamma\pi^+\pi^-\pi^+\pi^-$

Listed here are formulae used in refs. [8,14]:

$$\langle\gamma 0^{-+}|\rho\rho\rangle = S_{\mu\nu}\epsilon_{\gamma\delta\lambda\sigma}p_1^\gamma p_2^\delta p_3^\lambda p_4^\sigma B_1(Q_{\psi\gamma X}) \cdot\left[f^{(\rho)}_{(12)}f^{(\rho)}_{(34)}B_1(Q_{\rho 12})B_1(Q_{\rho 34})B_1(Q_{X(12)(34)}) - f^{(\rho)}_{(14)}f^{(\rho)}_{(32)}B_1(Q_{\rho 14})B_1(Q_{\rho 32})B_1(Q_{X(14)(32)})\right], \quad (143)$$

$$\langle\gamma 0^{++}|\sigma\sigma\rangle = g_{\mu\nu}\left[f^{(\sigma)}_{(12)}f^{(\sigma)}_{(34)} + f^{(\sigma)}_{(14)}f^{(\sigma)}_{(32)}\right], \quad (144)$$

$$\langle\gamma 0^{++}|\rho\rho\rangle = g_{\mu\nu}\left[f^{(\rho)}_{(12)}f^{(\rho)}_{(34)}(q_{(12)}\cdot q_{(34)})B_1(Q_{\rho 12})B_1(Q_{\rho 34}) + f^{(\rho)}_{(14)}f^{(\rho)}_{(32)}(q_{(14)}\cdot q_{(32)})B_1(Q_{\rho 14})B_1(Q_{\rho 32})\right], \quad (145)$$

$$\langle\gamma 0^{++}|\pi\pi'(\pi\sigma)\rangle = g_{\mu\nu}\left[f^{(\pi')}_{(123)}(f^{(\sigma)}_{(12)} + f^{(\sigma)}_{(32)}) + f^{(\pi')}_{(234)}(f^{(\sigma)}_{(23)} + f^{(\sigma)}_{(34)}) + f^{(\pi')}_{(143)}(f^{(\sigma)}_{(14)} + f^{(\sigma)}_{(34)}) + f^{(\pi')}_{(214)}(f^{(\sigma)}_{(21)} + f^{(\sigma)}_{(14)})\right], \quad (146)$$

$$\langle\gamma 0^{++}|\pi\pi'(\pi\rho)\rangle = g_{\mu\nu}\left[f^{(\pi')}_{(123)}f^{(\rho)}_{(12)}q_{(12)\alpha}(p_3 - p_{(12)})^\alpha \cdot B_1(Q_{\pi'\rho 3})B_1(Q_{\rho 12}) + f^{(\pi')}_{(234)}f^{(\rho)}_{(23)}q_{(23)\gamma}(p_4 - p_{(23)})^\gamma \cdot B_1(Q_{\pi'\rho 4})B_1(Q_{\rho 23}) + \{1\leftrightarrow 3\} + \{2\leftrightarrow 4\} + \{1\leftrightarrow 3\ \&\ 2\leftrightarrow 4\}\right], \quad (147)$$

$$\langle\gamma 0^{++}|\pi a_1(\pi\rho)\rangle = g_{\mu\nu}\left[P^{(1)}_{\alpha\beta}(p_{(123)})p_4^\alpha q^\beta_{(12)}f^{(a_1)}_{(123)}f^{(\rho)}_{(12)} \cdot B_1(Q_{Xa_14})B_1(Q_{\rho 12}) + P^{(1)}_{\alpha\beta}(p_{(234)})p_1^\alpha q^\beta_{(23)}f^{(a_1)}_{(234)}f^{(\rho)}_{(23)} \cdot B_1(Q_{Xa_11})B_1(Q_{\rho 23}) + \{1\leftrightarrow 3\} + \{2\leftrightarrow 4\} + \{1\leftrightarrow 3\ \&\ 2\leftrightarrow 4\}\right], \quad (148)$$

$$\langle\gamma 2^{++}|(yy)1\rangle = X^{(yy)}_{\mu\nu}, \quad (149)$$

$$\langle\gamma 2^{++}|(yy)2\rangle = g_{\mu\nu}p^\alpha_\psi p^\beta_\psi X^{(yy)}_{\alpha\beta}B_2(Q_{\psi X\gamma}), \quad (150)$$

$$\langle\gamma 2^{++}|(yy)3\rangle = q_\mu X^{yy}_{\nu\alpha}p^\alpha_\psi B_2(Q_{\psi X\gamma}), \quad (151)$$

$$\langle\gamma 2^{++}|(f_2\sigma)1\rangle = P^{(2)}_{\mu\nu\alpha\beta}(K)\tilde{t}^{\alpha\beta}_{(12)}f^{(f_2)}_{(12)}f^{(\sigma)}_{(34)} + \{1\leftrightarrow 3\} + \{2\leftrightarrow 4\} + \{1\leftrightarrow 3\ \&\ 2\leftrightarrow 4\}, \quad (152)$$

$$\langle\gamma 2^{++}|(f_2\sigma)2\rangle = g_{\mu\nu}p^\alpha_\psi p^\beta_\psi P^{(2)}_{\alpha\beta\gamma\delta}(K)\tilde{t}^{(2)\gamma\delta}_{(12)}f^{(f_2)}_{(12)}f^{(\sigma)}_{(34)} \cdot B_2(Q_{\psi X\gamma}) + \{1\leftrightarrow 3\} + \{2\leftrightarrow 4\} + \{1\leftrightarrow 3\ \&\ 2\leftrightarrow 4\}, \quad (153)$$

$$\langle\gamma 2^{++}|(f_2\sigma)3\rangle = q_\mu p^\beta_\psi P^{(2)}_{\nu\beta\gamma\delta}(K)\tilde{t}^{(2)\gamma\delta}_{(12)}f^{(f_2)}_{(12)}f^{(\sigma)}_{(34)} \cdot B_2(Q_{\psi X\gamma}) + \{1\leftrightarrow 3\} + \{2\leftrightarrow 4\} + \{1\leftrightarrow 3\ \&\ 2\leftrightarrow 4\}. \quad (154)$$

The amplitudes involving X-particles of $J^P = 2^{++}$ involve a rank-two tensor, $X^{(yy)}$. The definition of this is given below:

$$X^{(\sigma\sigma)}_{\mu\nu} = f^{(\sigma)}_{(12)}f^{(\sigma)}_{(34)}B_2(Q_{X(12)(34)})P^{(2)}_{\mu\nu\alpha\beta}(K) \cdot(p^\alpha_{(12)} - p^\alpha_{(34)})(p^\beta_{(12)} - p^\beta_{(34)}) + \{2\leftrightarrow 4\}, \quad (155)$$

$$X^{(\rho\rho)}_{\mu\nu} = f^{(\rho)}_{(12)}f^{(\rho)}_{(34)}B_1(Q_{\rho 12})B_1(Q_{\rho 34}) \cdot P^{(2)}_{\mu\nu\alpha\beta}(K)\tilde{t}^{(1)\alpha}_{(12)}\tilde{t}^{(1)\beta}_{(34)} + \{2\leftrightarrow 4\}, \quad (156)$$

where the $L = 2$ decay for $X \to \rho\rho$ is ignored in view of the centrifugal barrier suppression.

From the flux tube model for hybrids, 1^{-+} hybrids with $I = 0$ decay dominantly into 4π through $a_1\pi$. Then $\psi \to \gamma\pi^+\pi^-\pi^+\pi^-$ is an ideal place for finding 1^{-+} hybrids. With high-statistics data at BES and CLEO-C, one should look for the isoscalar 1^{-+} hybrid in this channel. Here we add the formulae for the 1^{-+} hybrid production:

$$\langle\gamma 1^{-+}|\left[\pi a_1(\rho\pi)\right]1\rangle = g_{\mu\nu}p^\alpha_\psi P^{(1)}_{\alpha\beta}(K) \cdot\left[P^{(1)\beta\gamma}(p_{(123)})\tilde{t}^{(1)}_{(12)\gamma}f^{(a_1)}_{(123)}f^{(\rho)}_{(12)} + P^{(1)\beta\gamma}(p_{(234)})\tilde{t}^{(1)}_{(23)\gamma}f^{(a_1)}_{(234)}f^{(\rho)}_{(23)}\right] + \{1\leftrightarrow 3\} + \{2\leftrightarrow 4\} + \{1\leftrightarrow 3\ \&\ 2\leftrightarrow 4\}, \quad (157)$$

$$\langle\gamma 1^{-+}|\left[\pi a_1(\rho\pi)\right]2\rangle = q_\mu P^{(1)}_{\nu\beta}(K) \cdot\left[P^{(1)\beta\gamma}(p_{(123)})\tilde{t}^{(1)}_{(12)\gamma}f^{(a_1)}_{(123)}f^{(\rho)}_{(12)} + P^{(1)\beta\gamma}(p_{(234)})\tilde{t}^{(1)}_{(23)\gamma}f^{(a_1)}_{(234)}f^{(\rho)}_{(23)}\right] + \{1\leftrightarrow 3\} + \{2\leftrightarrow 4\} + \{1\leftrightarrow 3\ \&\ 2\leftrightarrow 4\}. \quad (158)$$

3.5 $\psi \to \gamma K^+K^-\pi^+\pi^-$

We construct the amplitudes $U^i_{\mu\nu}$ with a notation similar to that in the previous subsection for the $\psi \to \gamma\pi^+\pi^-\pi^+\pi^-$ channel. Here we denote K^+, K^-, π^+, π^- as 1, 2, 3, 4:

$$\langle\gamma 0^{-+}|K^*\bar{K}^*\rangle = S_{\mu\nu}\epsilon_{\gamma\delta\lambda\sigma}p_1^\gamma p_2^\delta p_3^\lambda p_4^\sigma f^{(K^*)}_{(14)}f^{(\bar{K}^*)}_{(23)} B_1(Q_{\psi\gamma X})B_1(Q_{K^*14}) \cdot B_1(Q_{\bar{K}^*23})B_1(Q_{X(14)(23)}), \quad (159)$$

$$\langle\gamma 0^{++}|\kappa\kappa\rangle = g_{\mu\nu}f^{(\kappa)}_{(14)}f^{(\kappa)}_{(23)}, \quad (160)$$

$$\langle\gamma 0^{++}|K^*\bar{K}^*\rangle = g_{\mu\nu}\tilde{t}^{(1)\alpha}_{(14)}\tilde{t}^{(1)}_{(23)\alpha}f^{(K^*)}_{(14)}f^{(K^*)}_{(23)}, \quad (161)$$

$$\langle\gamma 0^{++}|K^*\kappa\rangle = g_{\mu\nu}[\tilde{t}^{(1)}_{(K^*\bar{\kappa})\alpha}\tilde{t}^{(1)\alpha}_{(14)}f^{(K^*)}_{(14)}f^{(\kappa)}_{(23)} + \tilde{t}^{(1)}_{(\bar{K}^*\kappa)\alpha}\tilde{t}^{(1)\alpha}_{(23)}f^{(K^*)}_{(23)}f^{(\kappa)}_{(14)}], \quad (162)$$

$$\langle\gamma 1^{++}|(K^*\bar{K}^*)1\rangle = \epsilon_{\mu\nu\lambda\alpha}p_\psi^\lambda\epsilon^{\alpha\beta\gamma\delta}K_\beta \cdot\tilde{t}^{(1)}_{(14)\gamma}\tilde{t}^{(1)}_{(23)\delta}f^{(K^*)}_{(14)}f^{(\bar{K}^*)}_{(23)}, \quad (163)$$

$$\langle\gamma 1^{++}|(K^*\bar{K}^*)2\rangle = q_\mu S_{\nu\alpha}\epsilon^{\alpha\beta\gamma\delta}K_\beta\tilde{t}^{(1)}_{(14)\gamma}\tilde{t}^{(1)}_{(23)\delta} \cdot f^{(K^*)}_{(14)}f^{(\bar{K}^*)}_{(23)}B_2(Q_{\psi\gamma X}), \quad (164)$$

$$\langle\gamma 1^{++}|(K^*\kappa)1\rangle = \epsilon_{\mu\nu\lambda\alpha}p_\psi^\lambda\epsilon^{\alpha\beta\gamma\delta}p_{X\beta} \cdot\Big[\tilde{t}^{(1)}_{(K^*\bar{\kappa})\gamma}\tilde{t}^{(1)}_{(14)\delta}f^{(K^*)}_{(14)}f^{(\kappa)}_{(23)} +\tilde{t}^{(1)}_{(\bar{K}^*\kappa)\gamma}\tilde{t}^{(1)}_{(23)\delta}f^{(K^*)}_{(23)}f^{(\kappa)}_{(14)}\Big], \quad (165)$$

$$\langle\gamma 1^{++}|(K^*\kappa)2\rangle = q_\mu S_{\nu\alpha}\epsilon^{\alpha\beta\gamma\delta}p_{X\beta} \cdot\Big[\tilde{t}^{(1)}_{(K^*\bar{\kappa})\gamma}\tilde{t}^{(1)}_{(14)\delta}f^{(K^*)}_{(14)}f^{(\kappa)}_{(23)} + \tilde{t}^{(1)}_{(\bar{K}^*\kappa)\gamma} \cdot\tilde{t}^{(1)}_{(23)\delta}f^{(K^*)}_{(23)}f^{(\kappa)}_{(14)}\Big]B_2(Q_{\psi\gamma X}), \quad (166)$$

$$\langle\gamma 2^{++}|(yz)1\rangle = X^{(yz)}_{\mu\nu}, \quad (167)$$

$$\langle\gamma 2^{++}|(yz)2\rangle = g_{\mu\nu}p_\psi^\alpha p_\psi^\beta X^{(yz)}_{\alpha\beta}B_2(Q_{\psi\gamma X}), \quad (168)$$

$$\langle\gamma 2^{++}|(yz)3\rangle = q_\mu p_\psi^\alpha X^{(yz)}_{\alpha\nu}B_2(Q_{\psi\gamma X}), \quad (169)$$

$$\langle\gamma 4^{++}|(yy)1\rangle = Z^{(yy)}_{\mu\nu\lambda\sigma}p_\psi^\lambda p_\psi^\sigma B_2(Q_{\psi\gamma X}), \quad (170)$$

$$\langle\gamma 4^{++}|(yy)2\rangle = g_{\mu\nu}p_\psi^\alpha p_\psi^\beta p_\psi^\gamma p_\psi^\delta Z^{(yy)}_{\alpha\beta\gamma\delta}B_4(Q_{\psi\gamma X}), \quad (171)$$

$$\langle\gamma 4^{++}|(yy)3\rangle = q_\mu Z^{(yy)}_{\nu\lambda\sigma\alpha}p_\psi^\lambda p_\psi^\sigma p_\psi^\alpha B_4(Q_{\psi\gamma X}), \quad (172)$$

$$\langle\gamma 2^{-+}|(K^*\bar{K}^*)1A\rangle = \epsilon_{\mu\nu\alpha\beta}p_\psi^\alpha A^{\beta\lambda}p_{\psi\lambda}B_1(Q_{\psi\gamma X}), \quad (173)$$

$$\langle\gamma 2^{-+}|(K^*\bar{K}^*)1B\rangle = \epsilon_{\mu\nu\alpha\beta}p_\psi^\alpha B^{\beta\lambda}p_{\psi\lambda}B_1(Q_{\psi\gamma X}), \quad (174)$$

$$\langle\gamma 2^{-+}|(K^*\bar{K}^*)2A\rangle = S_{\mu\nu}A^{\lambda\sigma}p_{\psi\lambda}p_{\psi\sigma}B_3(Q_{\psi\gamma X}), \quad (175)$$

$$\langle\gamma 2^{-+}|(K^*\bar{K}^*)2B\rangle = S_{\mu\nu}B^{\lambda\sigma}p_{\psi\lambda}p_{\psi\sigma}B_3(Q_{\psi\gamma X}), \quad (176)$$

$$\langle\gamma 2^{-+}|(K^*\bar{K}^*)3A\rangle = q_\mu S_{\nu\gamma}A^{\gamma\delta}p_{\psi\delta}B_3(Q_{\psi\gamma X}), \quad (177)$$

$$\langle\gamma 2^{-+}|(K^*\bar{K}^*)3B\rangle = q_\mu\epsilon_{\nu\gamma}B^{\gamma\delta}p_{\psi\delta}B_3(Q_{\psi\gamma X}) \quad (178)$$

with $S_{\mu\nu}$ defined as in eq. (107). The amplitudes involving X-particles of $J^P = 2^+$ involve a rank-two tensor, $X^{(yz)}$. The definition of this is given below:

$$X^{(\kappa\kappa)}_{\mu\nu} = \tilde{t}^{(2)}_{(\kappa\bar{\kappa})\mu\nu}f^{(\kappa)}_{(14)}f^{(\kappa)}_{(23)}, \quad (179)$$

$$X^{(K^*\bar{K}^*)}_{\mu\nu} = P^{(2)}_{\mu\nu\alpha\beta}(K)\tilde{t}^{(1)\alpha}_{(14)}\tilde{t}^{(1)\beta}_{(23)}f^{(K^*)}_{(14)}f^{(\bar{K}^*)}_{(23)}, \quad (180)$$

$$X^{(K^*\kappa)}_{\mu\nu} = P^{(2)}_{\mu\nu\alpha\beta}(K)\Big[\tilde{t}^{(1)\alpha}_{(K^*\bar{\kappa})}\tilde{t}^{(1)\beta}_{(14)}f^{(K^*)}_{(14)}f^{(\kappa)}_{(23)} + \tilde{t}^{(1)\alpha}_{(\bar{K}^*\kappa)}\tilde{t}^{(1)\beta}_{(23)}f^{(K^*)}_{(23)}f^{(\kappa)}_{(14)}\Big], \quad (181)$$

$$Z^{(\kappa\kappa)}_{\alpha\beta\gamma\delta} = \tilde{t}^{(4)}_{(\kappa\bar{\kappa})\alpha\beta\gamma\delta}f^{(\kappa)}_{(14)}f^{(\kappa)}_{(23)}, \quad (182)$$

$$Z^{(K^*\bar{K}^*)}_{\alpha\beta\gamma\delta} = P^{(4)}_{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'}(K)\tilde{t}^{(2)\alpha'\beta'}_{(K^*\bar{K}^*)} \cdot\tilde{t}^{(1)\gamma'}_{(14)}\tilde{t}^{(1)\delta'}_{(23)}f^{(K^*)}_{(14)}f^{(\bar{K}^*)}_{(23)}, \quad (183)$$

$$A_{\mu\nu} = P^{(2)}_{\mu\nu\alpha\alpha'}(K)\epsilon^{\alpha\beta\gamma\delta}K_\beta\tilde{t}^{(1)}_{(14)\gamma}\tilde{t}^{(1)}_{(23)\delta} \cdot\tilde{t}^{(1)\alpha'}_{(K^*\bar{K}^*)}f^{(K^*)}_{(14)}f^{(\bar{K}^*)}_{(23)}, \quad (184)$$

$$B_{\mu\nu} = P^{(2)}_{\mu\nu\alpha\alpha'}(K)\epsilon^{\alpha\beta\gamma\delta}K_\beta\tilde{t}^{(1)}_{(K^*\bar{K}^*)\gamma} \cdot\Big(\tilde{t}^{(1)}_{(14)\delta}\tilde{t}^{(1)\alpha'}_{(23)} + \tilde{t}^{(1)}_{(23)\delta}\tilde{t}^{(1)\alpha'}_{(14)}\Big)f^{(K^*)}_{(14)}f^{(\bar{K}^*)}_{(23)}, \quad (185)$$

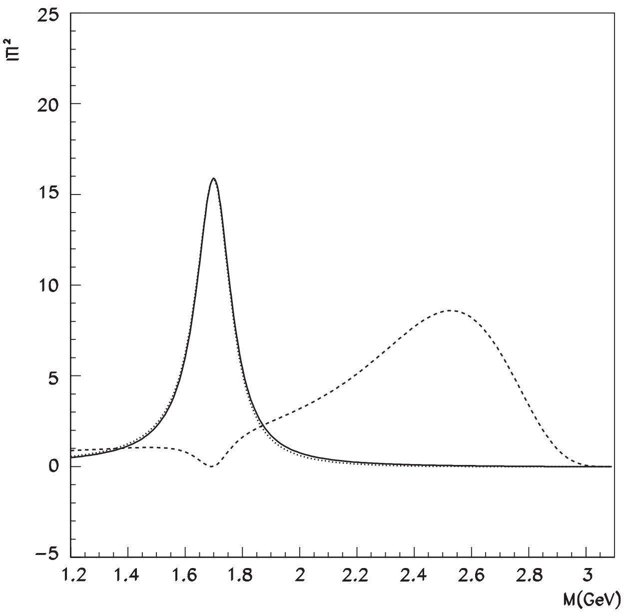

Fig. 1. Distortion on the Breit-Wigner amplitude squared by the s-dependence numerators: with $Q^2_{\psi\gamma f}B_2(Q_{\psi\gamma f})$ (solid line), with $30.3Q^4_{\psi\gamma f}B_4(Q_{\psi\gamma f})$ (dotted line), $20[Q^2_{\psi\gamma f}B_2(Q_{\psi\gamma f}) - 30.3Q^4_{\psi\gamma f}B_4(Q_{\psi\gamma f})]$ (dashed line).

where $A_{\mu\nu}$ corresponds to $2^{-+} \to K^*\bar{K}^*$ with $L = 1$ and $S = 1$, $B_{\mu\nu}$ corresponds to $2^{-+} \to K^*\bar{K}^*$ with $L = 1$ and $S = 2$. We ignore $2^{-+} \to K^*\bar{K}^*$ with $L = 3$ due to a strong centrifugal barrier.

4 Discussion

Here we add some points of general technique in fitting the data. The first concerns the fact that tensor amplitudes are not always unique. As an example, in $J/\psi \to \gamma f_2$, there are three independent helicity amplitudes. But the general formalism allows one to write down five covariant tensor amplitudes. Those five are independent in the process $J/\psi \to \omega f_2$, but for the radiative decay, gauge invariance makes two of them dependent on the other three. Two further linear combinations differ from the first three only by a different s-dependence arising from the momentum dependence built into the tensor expressions. Chung [11] recommends using all five combinations, so as to retain the differences in the possible s-dependence. However, this gives rise to a practical problem.

One is usually fitting resonances such as f_2 to the data. If two of the amplitudes differ from the others only in the s-dependence, this is equivalent to putting into the numerator of an f_2 Breit-Wigner amplitude a linear combination of two s-dependent terms with two free parameters. This may lead to a zero amplitude at the resonance mass and can give rise to a structure which may lie 500 MeV or 1 GeV away from f_2; it may be easily confused with the effects of other resonances. This is illustrated in fig. 1 for the amplitude squared $|T|^2$ taking as an example $J/\psi \to \gamma f_2(1700) \to \gamma K\bar{K}$. For the solid line, we use $T = Q^2_{\psi\gamma f} B_2(Q_{\psi\gamma f})/(M_f^2 - s - iM_f\Gamma_f)$ with $M_f = 1.7$ GeV and $\Gamma_f = 0.15$ GeV; for the dotted line which lies very close to the solid line, we use $T = 30.3 \cdot Q^4_{\psi\gamma f} B_4(Q_{\psi\gamma f})/(M_f^2 - s - iM_f\Gamma_f)$. The two different s-dependence numerators give a hardly visible difference in the line-shape of f_2. But if one allows two s-dependent terms in the numerator with two free parameters, a ridiculous shape (dashed line) could happen for a single resonance $f_2(1700)$; in this illustration we use $20[Q^2_{\psi\gamma f} B_2(Q_{\psi\gamma f}) - 30.3 Q^4_{\psi\gamma f} B_4(Q_{\psi\gamma f})]$ in the numerator. Although theoretically this possibility cannot be excluded, it is very odd and in practice one may end up fitting other 2^{++} components far away from the f_2-resonance mass with f_2. One therefore should be very careful in drawing conclusions from a fit using more than the minimum number of amplitudes with different angular dependence.

In J/ψ radiative decays, the $c\bar{c}$ pair annihilates to gluons. This requires a short-range interaction with a range of order $1/m_c$, where m_c is the mass of the c-quark. Therefore the centrifugal barrier for $J/\psi \to \gamma X$ is strong. Some production with $L = 1$ is observed (at momentum transfer ≤ 1 GeV/c), but we find little evidence for $L > 1$.

We thank the Royal Society for funds allowing a collaboration between Queen Mary College, University of London and IHEP, Beijing. We thank V.V. Anisovich and A.V. Sarantsev for many helpful discussions. A small part of the formulae presented here were independently written and cross-checked by them. We thank L.Y. Dong, Z.J. Guo and J.L. Hu for the Monte Carlo simulation of some formulae in this paper and for pointing out some typos. The work is partly supported by CAS Knowledge Innovation Project (KJCX2-SW-N02) and the National Natural Science Foundation of China under Grant no. 10225525.

References

1. R.A. Briere *et al.*, Preprint CLNS-01-1742.
2. V.V. Anisovich, D.V. Bugg, A.V. Sarantsev, Nucl. Phys. A **537**, 501 (1992); V.V. Anisovich *et al.*, Nucl. Phys. A **544**, 747 (1992).
3. A.V. Anisovich *et al.*, Phys. Lett. B **452**, 173; 180; 187 (1999); **468**, 304; 309 (1999); **500**, 222 (2001); **507**, 23 (2001).
4. A.V. Anisovich, V.A. Sadovnikova, Eur. Phys. J. A **2**, 199 (1998); A.V. Anisovich *et al.*, J. Phys. G **28**, 15 (2002).
5. BES Collaboration, Phys. Lett. B **440**, 217 (1998).
6. BES Collaboration, Phys. Lett. B **446**, 356 (1999).
7. BES Collaboration, Phys. Lett. B **472**, 200 (2000).
8. BES Collaboration, Phys. Lett. B **472**, 208 (2000).
9. BES Collaboration, Phys. Lett. B **476**, 25 (2000).
10. W. Rarita, J. Schwinger, Phys. Rev. **60**, 61 (1941); E. Behtends, C. Fronsdal, Phys. Rev. **106**, 345 (1957).
11. S.U. Chung, Phys. Rev. D **48**, 1225 (1993); **57**, 431 (1998).
12. F. von Hippel, C. Quigg, Phys. Rev. D **5**, 624 (1972).
13. W. Greiner *et al.*, *Quantum Chromodynamics* (Springer, Heidelberg, 2002) p. 242.
14. D.V. Bugg *et al.*, Phys. Lett. B **353**, 378 (1995).